实分析期中考查试题

截止时间：11月5日10时

In the following exercises m always denote the Lebesgue measure, and m^* the Lebesgue outer measure.

习题1. Fix a prime number p. We define a function on $\mathbb{Z} \times \mathbb{Z}$ by $d(x, x) = 0$ and $d(x, y) = p^{-n}$ for $x \neq y$ where n is the largest integer such that p^n divides $y - x$. Prove that $(\mathbb{Z}, d)$ is a metric space. Is $(\mathbb{Z}, d)$ compact?

习题2. Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R} : x \in E_k, \text{ for infinitely many } k\} = \limsup_{k \to \infty}(E_k).$$

1. Show that E is measurable.

2. Prove $m(E) = 0$.

习题3. Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]} \text{ almost everywhere.}$$

习题4. Suppose E is measurable with $m(E) < \infty$, and

$$E = E_1 \cup E_2, \; E_1 \cap E_2 = \emptyset.$$

If $m(E) = m^*(E_1) + m^*(E_2)$, then E_1 and E_2 are measurable.

习题5. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \ a.e.x.$$

习题6. Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

习题7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is on-decreasing. Show that if $A \subset \mathbb{R}$ is a Borel set, then so is $f(A)$.

习题8. Consider the curve $\Gamma = \{y = f(x)\}$ in $\mathbb{R}^2$, $0 \le x \le 1$. Assume that f is twice continuously differentiable in $0 \le x \le 1$. Then show that $m(\Gamma + \Gamma) > 0$ if and only if $\Gamma + \Gamma$ contains an open set, if and only if f is not linear.