Lecture 21 Ergodicity and Mixing

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Outline

Review

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Essentially invariant function

Definition

Let T be a measure-preserving transformation (or a flow) on a measure space $(X, \mathcal{X}, \mu)$. A measurable function $f : X \to \mathbb{R}$ is *essentially T-invariant* if we have $\mu(\{x \in X : f(T^t x) \neq f(x)\}) = 0$ for every t.

Essentially invariant set

Definition

Let T be a measure-preserving transformation (or a flow) on a measure space $(X, \mathcal{X}, \mu)$. A measurable set A is *essentially* T*-invariant* if its characteristic function 1_A is essentially T-invariant, equivalently, if $\mu(T^{-1}(A) \Delta A) = 0$.

Ergodicity

Definition

Let T be a measure-preserving transformation (or a flow) on a measure space $(X, \mathcal{X}, \mu)$. T is called ergodic if any essentially T-invariant measurable set has either measure 0 or full measure. Equivalently, T is ergodic if any essentially T-invariant measurable function is constant mod 0.

Proposition

Let T be a measure-preserving transformation (or a flow) on a finite measure space $(X, \mathcal{X}, \mu)$, and let $p \in (0, +\infty]$. Then T is ergodic if and only if every essentially invariant function $f \in L^p(X, \mu)$ is constant mod 0.

Proof.

If T is ergodic, then every essentially T-invariant measurable function is constant mod 0.
To prove the converse, let f be an essentially invariant measurable function on X. Then every $M > 0$, the function

$$f_M(x) = \begin{cases} f(x) & \text{if } f(x) \leq M, \\ 0 & \text{if } f(x) > M \end{cases}$$

is bounded and so belongs to $L^p(X, \mu)$. It is also essentially invariant. Therefore it is constant mod 0. Since this is true for any M, it follows that f itself is constant mod 0. □

Essentially invariant function and strictly invariant function

Proposition

Let T be a measure-preserving transformation (or a flow) on a measure space $(X, \mathcal{X}, \mu)$, and suppose that $f : X \to \mathbb{R}$ is essentially invariant for T. Then there is a strictly invariant measurable function $\tilde{f}$ such that $f(x) = \tilde{f}(x)$ mod 0.

We shall prove the proposition for a measurable flow.

Proof.

Consider the measurable map $\Phi : X \times \mathbb{R} \to \mathbb{R}$, $\Phi(x,t) = f(T^t x) - f(x)$, and the product measure $\nu = \mu \times \lambda$ in $X \times \mathbb{R}$, where λ is the Lebesgue measure on $\mathbb{R}$. The set $A = \Phi^{-1}(0)$ is a measurable subset of $X \times \mathbb{R}$. Since f is essentially T-invariant, for each $t \in \mathbb{R}$ the set

$$A_t = \{(x,t) \in X \times \mathbb{R} : f(T^t x) = f(x)\}$$

has full μ-measurable in $X \times \{t\}$. By the Fubini theorem, the set

$$A_f = \{x \in X : f(T^t x) = f(x) \text{ for a.e. } t \in \mathbb{R}\}$$

has full μ-measurable in X. □

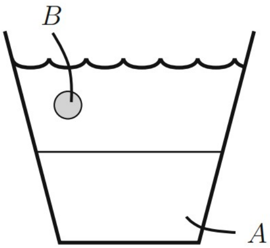
Proof.

Set

$$\tilde{f}(x) = \begin{cases} f(y) & \text{if } T^t x = y \in A_f \text{ for some } t \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

If $T^t x = y \in A_f$ and $T^s x = z \in A_f$, then y and z lie on the same orbit, and the value of f along this orbit is equal almost everywhere to $f(y)$ and $f(z)$, so $f(y) = f(z)$. Therefore $\tilde{f}$ is well defined and strictly T-invariant. □

Outline



Mixing property

Strong mixing

Definition

A measure-preserving transformation (or flow) T on a probability space $(X, \mathcal{X}, \mu)$ is called *(strong) mixing* if

$$\lim_{t\to\infty} \mu(T^{-t}(A) \cap B) = \mu(A) \cdot \mu(B)$$

for any two measurable sets $A,\ B \in \mathcal{X}$.

Proposition

T is mixing if and only if

$$\lim_{t\to\infty} \int_X f(T^t(x)) \cdot g(x) d\mu = \int_X f(x) d\mu \cdot \int_X g(x) d\mu$$

for any bounded measurable functions f, g.

Weak mixing

Definition

A measure-preserving transformation T on a probability space $(X, \mathcal{X}, \mu)$ is called *weak mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0$$

for any two measurable sets $A,\ B \in \mathcal{X}$.

Equivalently, T is weak mixing if and only if for all bounded measurable functions f, g,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\int_X f(T^i(x))g(x)d\mu - \int_X f d\mu \int_X g d\mu| = 0.$$

Definition

A measure-preserving flow T^t on a probability space $(X, \mathcal{X}, \mu)$ is called *weak mixing* if

$$\lim_{n\to\infty} \frac{1}{t} \int_0^t |\mu(T^{-s}(A) \cap B) - \mu(A)\mu(B)| ds = 0$$

for any two measurable sets $A,\ B \in \mathcal{X}$.

Equivalently, T^t is weak mixing if and only if for all bounded measurable functions f, g,

$$\lim_{n\to\infty} \frac{1}{t} \int_0^t | \int_X f(T^s(x))g(x)d\mu - \int_X f d\mu \int_X g d\mu| ds = 0.$$

Result

Proposition

Mixing implies weak mixing, and weak mixing implies ergodicity.

Proposition

Let X be a compact metric space, $T : X \to X$ a continuous map, and μ a T-invariant Borel measure on X.

1. *If T is ergodic, then the orbit of μ-almost every point is dense in suppμ.*
2. *If T is mixing, then T is topologically mixing on suppμ.*

Circle rotation

Proposition

The circle rotation R_α is ergodic with respect to Lebesgue measure if and only if α is irrational.

Proof.

Suppose α is irrational. It is enough to prove that any bounded R_α-invariant function $f : S^1 \to \mathbb{R}$ is constant mod 0. Since $f \in L^2(S^1, \lambda)$, the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{2n\pi i x}$ of f converges to f in the L^2 norm.

The series $\sum_{n=-\infty}^{\infty} a_n e^{2n\pi i(x+\alpha)}$ converges to $f \circ R_\alpha$. Since $f = f \circ R_\alpha$ mod 0, by the uniqueness of Fourier coefficients we have that $a_n = a_n e^{2n\pi i\alpha}$ for all $n \in \mathbb{Z}$. Since $e^{2n\pi i\alpha} \neq 1$ for $n \neq 0$, we conclude that $a_n = 0$ for $n \neq 0$, so f is constant mod 0.

Circle rotation

Proposition

The circle rotation R_α is ergodic with respect to Lebesgue measure if and only if α is irrational.

Proof.

Suppose α is rational, then we may write $\alpha = \frac{p}{q}$ in the lowest terms, so that $R_\alpha^q = I_{\mathbb{T}}$ is the identity map. Pick any measurable set $A \subset S^1$ with $\lambda(A) \in (0, \frac{1}{q})$. Then

$$B = A \cup R_\alpha A \cup \cdots \cup R_\alpha^{q-1} A$$

is a measurable set invariant under R_α with $\alpha(B) \in (0, 1)$, which implies that R_α is not ergodic. □

Expanding endomorphism of the circle

Proposition

An expanding endomorphism $E_m : S^1 \to S^1$ *is mixing with respect to Lebesgue measure.*

Proof.

Since any measurable subset of S^1 can be approximated by a finite union of intervals, it is sufficient to consider two intervals $A = [p/m^i, (p+1)/m^i],\ p \in \{0, \cdots, m^i - 1\}$, and $B = [q/m^i, (q+1)/m^j]$, $q \in \{0, \cdots, m^j - 1\}$. Recall that

$$E_m^{-1}(B) = \cup_{k=0}^{m-1} [(km^j + q)/m^{j+1}, (km^j + q + 1)/m^{j+1}].$$

By induction we can show that $E_m^{-n}(B)$ is the union of m^n uniformly spaced intervals of length $1/m^{j+n}$. Thus for $n > i$, the intersection $A \cap E_m^{-n}(B)$ consists of m^{n-i} intervals of length $m^{-(n+j)}$. Thus

$$\lambda(A \cap E_m^{-n}(B)) = m^{n-i}(1/m^{n+j}) = m^{-i-j} = \lambda(A) \cdot \lambda(B).$$

So E_m is mixing. □

Hyperbolic toral automorphism

Proposition

Any hyperbolic toral automorphism $A : \mathbb{T}^n \to \mathbb{T}^n$ *is ergodic with respect to Lebesgue measure.*

Proof.

We consider here only the case

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2;$$

the argument in the general case is similar. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a bounded A-invariant measurable function. The Fourier series $\sum_{m,n=-\infty}^{\infty} a_{m,n} e^{2\pi i(mx+ny)}$ of f converges to f in L^2. The series

$$\sum_{m,n=-\infty}^{\infty} a_{m,n} e^{2\pi i(m(2x+y)+n(x+y))}$$

converges to $f \circ A$. Since f is invariant, uniqueness of Fourier coefficients implies that $a_{m,n} = a_{2m+n,m+n}$ for all m, n. Since A does not have eigenvalues on the unit circle, if $a_{m,n} \neq 0$ for some $(m.n) \neq (0,0)$, then $a_{i,j} = a_{m,n} \neq 0$ with arbitrarily large $|i| + |j|$, and the Fourier series diverges. □

Proposition

A toral automorphism of $\mathbb{T}^n$ corresponding to an integer matrix A is ergodic if and only if no eigenvalue of A is a root of unity.

Proposition

A hyperbolic toral automorphism is mixing.

Markov shift

- Let A be an $m \times m$ stochastic matrix, i.e., A has non-negative entries, and the sum of every row is 1.
- Suppose A has a non-negative left eigenvector q with egienvalue 1 and sum of entries equal to 1 (recall that if A is irreducible, then q exists and is unique).

We define a Borel probability measure $P = P_{A,q}$ on Σ_m (and Σ_m^+) as follows: for a cylinder C_j^n of length 1, we define $P(C_j^n) = q_j$; for a cylinder $C_{j_0,j_1,\cdots,j_k}^{n,n+1,\cdots,n+k} \subset \Sigma_m$ (or Σ_m^+) with $k+1>1$ consecutive indices,

$$P(C_{j_0,j_1,\cdots,j_k}^{n,n+1,\cdots,n+k}) = q_{j_0} \prod_{i=0}^{k-1} A_{j_i j_{i+1}}.$$

The pair (A,q) is called a Markov chain on the set $\{1,\cdots,m\}$.

- It can be shown that P extends uniquely to a shift-invariant σ-additive measure defined on the completion $\mathcal{C}$ of the Borel σ-algebra generated by the cylinders; it is called the *Markov measure* corresponding to A and q.
- The measure space $(X, \mathcal{C}, P)$ is a non-atomic Lebesgue probability space.
- If A is irreducible, this measure is uniquely determined by A.
- The shift σ on $(X, \mathcal{C}, P)$ is called a *Markov shift*.

A very important particular case of this situation arises when the transition probabilities do not depend on the initial state. In this case each row of A is the left eigenvector q, the shift-invariant measure P is called a *Bernoulli measure*, and the shift is called a *Bernoulli automorphism*.

Proposition

If A is a primitive stochastic $m \times m$ matrix, then the shift σ is mixing in Σ_m with respect to the Markov measure $P(A)$.

Mixing of order 3

A transformation preserving a probability measure is called mixing of order 3 if it satisfies the following property: for any measurable sets $A,\ B,\ C \subset X$,

$$\mu(A \cap T^{-n_1} B \cap T^{-n_1 - n_2} C) \to \mu(A)\mu(B)\mu(C).\ n_1, n_2 \to \infty.$$

One can generalize the above definition to mixing of higher orders.

Conjecture (Rohklin's problem on mixing systems)

Any mixing system is mixing of order 3.

B. Host proved that a mixing transformation whose spectrum is singular is mixing of all orders.