Lecture 5 Data processing inequality and Fano inequality

September 13th, 2022

Outline

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1. Exercises Review
2. Data processing inequality
3. Fano's inequality
4. Another inequality relating probability of error

Example

A $(7, 4)$ *Hamming code can correct any one error; might there be a* $(14, 8)$ *code that can correct any two errors?*

Proof.

When the decoder receives $\mathbf{r} = \mathbf{t} + \mathbf{n}$, his aim is to deduce both $\mathbf{t}$ and $\mathbf{n}$ from $\mathbf{r}$. If it is the case that the sender can select any transmission $\mathbf{t}$ from a code of size $S_{\mathbf{t}}$, and the channel can select any noise vector from a set of size $S_{\mathbf{n}}$, and those two selections can be recovered from the received bit string $\mathbf{r}$, which is one of at most 2^N possible strings, then it must be the case that

$$S_{\mathbf{t}} S_{\mathbf{n}} \leq 2^N.$$

So, for a (N, K) two-error-correcting code,

$$2^K [\binom{N}{2} + \binom{N}{1} + \binom{N}{0}] \leq 2^N.$$

however the inequality does not hold for $K = 8$ and $N = 14$, which rules out the possibility that there is a $(14, 8)$ code that is 2-error correcting. □

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Data processing inequality

Lemma

If $X \to Y \to Z$, *then*

$$I(X;Y) \geq I(X;Z).$$

Proof.

By the chain rule, we can expand mutual information in two different ways:

$$
\begin{aligned}
I(X;Y,Z) &= I(X;Z) + I(X;Y|Z) \\
&= I(X;Y) + I(X;Z|Y).
\end{aligned}
$$

Since X and Z are conditionally independent given Y, we have $I(X;Z|Y) = 0$. Since $I(X;Y|Z) \geq 0$, we have

$$I(X;Y) \geq I(X;Z).$$

The equality holds if and only if $I(X;Y|Z) = 0$ (i.e., $X \to Z \to Y$ forms a Markov chain). Similarly, one can prove that $I(Y;Z) \geq I(X;Z)$. □

Corollary

If $Z = g(Y)$, then $I(X;Y) \geq I(X;g(Y))$.

Corollary

If $X \to Y \to Z$, then $I(X;Y|Z) \leq I(X;Y)$.

Note that it is also possible that $I(X;Y|Z) > I(X;Y)$ when X, Y and Z do not form a Markov chain.

For example, let X and Y be independent fair binary random variables, and let $Z = X + Y$. Then $I(X;Y) = 0$, but $I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = H(X|Z) = P(Z = 1)P(X|Z = 1) = \frac{1}{2}$ bit.

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Fano's inequality

Theorem

Let X and Y be two random variables, correlated in general. with alphabet $\mathcal{X}$ and $\mathcal{Y}$, respectively, where $\mathcal{X}$ is finite but $\mathcal{Y}$ can be countably infinite. Let $\hat{X} := g(Y)$ be an estimate of X from observing Y, where $g : \mathcal{Y} \to \mathcal{X}$ is a given estimation function. Define the probability of error as

$$P_e := Pr[\hat{X} \neq X].$$

Then the following inequality holds

$$H(X|Y) \leq h_b(P_e) + P_e \cdot \log_2(|\mathcal{X}| - 1),$$

wehere $h_b(x) := -x \log_2 x - (1-x) \log_2(1-x)$ for $0 \leq x \leq 1$ is the binary entropy function.

Proof.

Define a new random variable,

$$E := \begin{cases} 1, & \text{if } g(Y) \neq X \\ 0, & \text{if } g(Y) = X. \end{cases}$$

Then using the chain rule for conditional entropy, we obtain

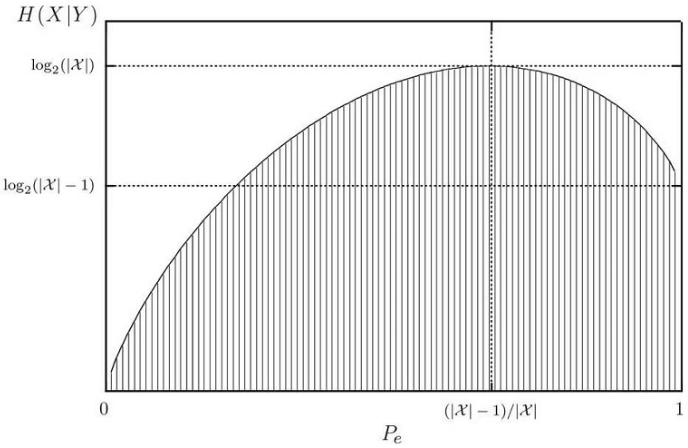
$$\begin{aligned} H(E, X|Y) &= H(X|Y) + H(E|X, Y) \\ &= H(E|Y) + H(X|E, Y). \end{aligned}$$

Observe that E is a function of X and Y; hence, $H(E|X, Y) = 0$. Since conditioning never increases entropy, $H(E|Y) \leq H(E) = h_b(P_e)$. The remaining term, $H(X|E, Y)$, can be bounded as follows:

$$\begin{aligned} H(E, X|Y) &= Pr[E = 0]H(X|Y, E = 0) + Pr[E = 1]H(X|Y, E = 1) \\ &\leq (1 - P_e) \cdot 0 + P_e \cdot \log_2(|\mathcal{X}| - 1), \end{aligned}$$

since $X = g(Y)$ for $E = 0$, and given $E = 1$, we can upper bound the conditional entropy by the logarithm of the number of remaining outcomes, i.e., $(|\mathcal{X}| - 1)$. Combining these results completes the proof. □

Permissible $(P_e, H(X|Y))$ region due to Fano's inequality



Fano's inequality yields upper and lower bounds on P_e in terms of $H(X|Y)$. This is illustrated in last page, where we plot the region for the pairs $(P_e, H(X|Y))$ that are permissible under Fano's inequality.

In the figure, the boundary of the permissible (dashed) region is given by the function

$$f(P_e) := h_b(P_e) + P_e \cdot \log_2(|\mathcal{X}| - 1).$$

We obtain that when

$$\log_2(|\mathcal{X}| - 1) \leq H(X|Y) \leq \log_2(|\mathcal{X}|),$$

P_e can be upper and lower bounded as follows:

$$0 < \inf\{a : f(a) \geq H(X|Y)\} \leq P_e \leq \sup\{a : f(a) \geq H(X|Y)\} < 1.$$

Furthermore, when

$$0 < H(X|Y) \leq \log_2(|\mathcal{X}| - 1),$$

only the lower bound holds:

$$P_e \geq \inf\{a : f(a) \geq H(X|Y)\} > 0.$$

Thus for all nonzero values of $H(X|Y)$, we obtain a lower bound (of the same form above) on P_e; the bound implies that if $H(X|Y)$ is bounded away from zero, P_e is also bounded away from zero.

Fano's inequality cannot be improved in the sense that the lower bound, $H(X|Y)$, can be achieved for some specific cases. Any bound that can be achieved in some cases is often referred to as sharp.

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From the proof of the above lemma, we can observe that equality holds in Fano's inequality, if $H(E|Y) = H(E)$ and $H(X|Y, E = 1) = \log_2(|\mathcal{X} - 1)$. The former is equivalent to E being independent of Y, and the latter holds iff $P_{X|Y}(\cdot|y)$ is uniformly distributed over the set $\mathcal{X} \backslash \{g(y)\}$. We can therefore create an example in which equality holds in Fano's inequality.

Example

Suppose that X and Y are two independent random variables which are both uniformly distributed on the alphabet $\{0, 1, 2\}$. Let the estimating function be given by $g(y) = y$. Then

$$P_e = Pr[g(Y) \neq X] = Pr[Y \neq X] = 1 - \sum_{x=0}^{2} P_X(x) P_Y(x) = \frac{2}{3}.$$

In this case, equality is achieved in Fano's inequality, i.e.,

$$h_b(\frac{2}{3}) + \frac{2}{3} \cdot \log_2(3-1) = H(X|Y) = H(X) = \log_2 3.$$

Outline

Let X and X' be two independent identically distributed random variables with entropy $H(X)$. The probability at $X = X'$ is given by

$$Pr(X = X') = \sum_x p^2(x).$$

Lemma

If X and X' are i.i.d. with entropy $H(X)$.

$$Pr(X = X') \geq 2^{-H(X)},$$

with equality if and only if X has a uniform distribution.

Proof.

Suppose that $X \sim p(x)$. By Jensen's inequality, we have

$$2^{E \log p(x)} \leq E 2^{\log p(x)},$$

which implies that

$$2^{-H(X)} = 2^{\sum p(x) \log p(x)} \leq \sum p(x) 2^{\log p(x)} = \sum p^2(x).$$

□

Corollary

Let X, X' *be independent with* $X \sim p(x)$, $X' \sim r(x)$, $x, x' \in \mathcal{X}$. *Then*

$$\begin{aligned} P(X = X') &\geq 2^{-H(p) - D(p\|r)}, \\ P(X = X') &\geq 2^{-H(r) - D(r\|p)}. \end{aligned}$$

Proof.

We have

$$\begin{aligned} 2^{-H(p)-D(p\|r)} &= 2^{\sum p(x)\log p(x)+\sum p(x)\log\frac{r(x)}{p(x)}} \\ &= 2^{\sum p(x)\log r(x)} \\ &\leq \sum p(x)2^{\log r(x)} \\ &= \sum p(x)r(x) \\ &= P(X=X'), \end{aligned}$$

where the inequality follows from Jensen's inequality and the convexity of the function $f(y)=2^y$. □