From Poincaré Recurrence to Ergodic Theorems

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Outline

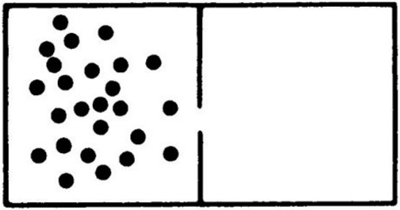
In mathematics and physics, the Poincaré recurrence theorem states that certain dynamical systems will, after a sufficiently long but finite time, return to a state arbitrarily close to (for continuous state systems), or exactly the same as (for discrete state systems), their initial state.

The Poincaré recurrence theorem

Theorem

Let $T : X \to X$ be a measure preserving transformation on a probability space $(X, \mathcal{X}, \mu)$, and let $E \subset X$ be a measurable set. Then almost every point $x \in E$ returns to E infinitely often. That is to say, there exists a measurable set $F \subset E$ with $\mu(F) = \mu(E)$ with the property that for every $x \in F$ there exist integers $0 < n_1 < n_2 < \cdots$ with $T^{n_i} x \in E$ for all $i \geq 1$.

If you open a partition separating a chamber containing gas and a chamber with a vacuum, then after a while the gas molecules will again collect in the first chamber.



Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure-preserving system, and let A be a measurable set with $\mu(A) > 0$. By Poincaré recurrence, the first return time to A, defined by

$$r_A(x) = \inf_{n \geq 1} \{n | T^n(x) \in A\}$$

exists (that is, finite) almost everywhere.

Definition

The map $T_A : A \to A$ defined (almost everywhere) by

$$T_A(x) = T^{r_A(x)}(x)$$

is called the transformation *induced* by T on the set A.

Lemma

The induced transformation T_A is a measure-preserving transformation on the space $(A, \mathcal{B}|_A, \mu_A = \frac{1}{\mu(A)}\mu|_A, T_A)$. If T is ergodic with respect to μ then T_A is ergodic with respect to μ_A.

Theorem (Kac)

Let $(X, \mathcal{B}, \mu, T)$ *be an ergodic measure-preserving system and let* $A \in \mathcal{B}$ *have* $\mu(A) > 0$. *Then the expected return time to* A *is* $\frac{1}{\mu(A)}$; *equivalently*

$$\int_A r_A d\mu = 1.$$

Theorem

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system, and let P_T denote the orthogonal projection onto the closed subspace

$$I = \{g \in L^2_\mu | U_T g = g\} \subset L^2_\mu$$

. Then for any $f \in L^2_\mu$,

$$\frac{1}{N} \sum_{n=0}^{N-1} U_T^n f \xrightarrow[L^2_\mu]{} P_T f$$

Theorem

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. If $f \in \mathscr{L}^1_\mu$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x)$$

converges almost everywhere and in $L^1(\mu)$ to a T-invariant function $f^* \in \mathscr{L}^1_\mu$, and

$$\int f^* d\mu = \int f d\mu.$$

If T is ergodic, then $f^*(x) = \int f d\mu$ almost everywhere.

Theorem (Szemerédi's theorem)

Any set in $\mathbb{Z}$ with positive upper Banach density contains arbitrarily long arithmetic progression.

Proposition (Furstenberg correspondence principle)

Assume that $E \subset \mathbb{Z}$ has positive upper Banach density. There exist a system (X, μ, T) and a subset A of X with $\mu(A) = d^*(E)$ and such that

$$d^*(E \cap (E + h_1) \cap \cdots \cap (E + h_k)) \geq \mu(A \cap T^{-h_1} A \cap \cdots \cap T^{-h_k} A)$$

for all $k \in \mathbb{N}$ and all $h_1, \cdots, h_k \in \mathbb{Z}$.

Theorem (Furstenberg multiple recurrence)

Let (X, μ, T) be a system and A be a subset of X with positive measure. Then for every $k \in \mathbb{N}$,

$$\liminf_{N\to\infty} \mathbb{E}_{n\in[N]}\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

Theorem (Walsh's polynomial multiple ergodic theorem)

Let $r, d, m \in \mathbb{N}$. If $T_1, \ldots, T_r$ are measure-preserving transformations of the probability space (X, μ) which spans a nilpotent group, $p_{i,j} : \mathbb{Z}^m \to \mathbb{Z}$ are polynomials, and $f_1, \ldots, f_d \in L^\infty(\mu)$, then for every Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ in $\mathbb{Z}^m$, the averages

$$\mathbb{E}_{\underline{n}\in\Phi_N}\Pi_{i=1}^{d}(\Pi_{j=1}^{r}T_j^{p_{i,j}(\underline{n})})f_i$$

convergence in $L^2(\mu)$.