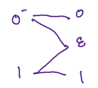
第25讲 习题课

2022年12月23日 星期五 07:58

率失真函数 $R(D)$

10.7. $X \sim \text{Bernoulli}(\frac{1}{2})$ $\Downarrow$ $H(X)=1$

$d(x,\hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}$



$\begin{bmatrix} 0 & 1 \\ \infty & 0 \end{bmatrix}$

$$p(x,\hat{x}) = \begin{array}{c|ccc|c} & 0 & \varepsilon & 1 & \\ 0 & \frac{1}{2}-\alpha & \alpha & 0 & x=0 \\ 1 & 0 & \beta & \frac{1}{2}-\beta & x=1 \end{array}$$

$$R(D) = \min_{\substack{\sum p(x,\hat{x})d(x,\hat{x}) \le D \\ p(x|\hat{x}):\ \sum p(x|\hat{x})p(x)d(x,\hat{x}) \le D}} I(X;\hat{X})$$

$I(X;\hat{X}) = H(X) - H(X|\hat{X})$

$= 1 - H(X|\hat{X})$

$= 1 - (\alpha+\beta) H(\frac{\alpha}{\alpha+\beta})$

$H(X|\hat{X})$

$= p(\hat{X}=0)\, H(X|\hat{X}=0)$ $=0$

$+ p(\hat{X}=1)\, H(X|\hat{X}=1)$ $=0$

$+ p(\hat{X}=\varepsilon)\, H(X|\hat{X}=\varepsilon)$

$= (\alpha+\beta) H(\frac{\alpha}{\alpha+\beta})$

限制: $\alpha+\beta \le D$

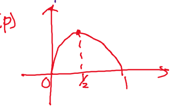
$\ge 1 - D\,H(\frac{\alpha}{\alpha+\beta})$

$H(\frac{\alpha}{\alpha+\beta}) \le H(\frac{1}{2}) = 1$ $\ge 1 - D\,H(\frac{1}{2}) = 1 - D$

当 $\alpha=\beta=D/2$，可取得 $(D\le 1)$

$D>1$ $I(X;\hat{X}) = \begin{cases} 1-D & D\le 1 \\ 0 & D>1 \end{cases}$

$H(p)$



KKT: $\min_{x\in Q} f(\vec{x})$

$Q := \{\vec{x} \in \mathcal{X}:\ g_i(\vec{x}) \le 0,\ 1\le i\le m \quad h_j(\vec{x})=0 \quad 1\le j\le l\}$

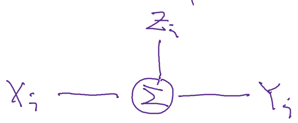
$\vec{x} = (x_1, \cdots, x_n)$ $\quad \vec{\lambda} = (\lambda_1, \cdots, \lambda_m)$ $\quad \vec{\nu} = (\nu_1, \cdots, \nu_l)$ $\quad$ KKT条件

$$\begin{cases} g_i(\vec{x}) \le 0 \quad \lambda_i \ge 0 \quad \lambda_i g_i(\vec{x}) = 0 & i=1,\cdots,m \\ h_j(\vec{x}) = 0 & j=1,\cdots,l \\ \frac{\partial f}{\partial x_k}(\vec{x}) + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_k}(\vec{x}) + \sum_{j=1}^{l} \nu_j \frac{\partial h_j}{\partial x_k}(\vec{x}) = 0 & k=1,\cdots,n \end{cases}$$

第9章 Gauss信道

$$C = \max_{f(x):\ EX^2\le P} I(X;Y) \qquad C = \max_{p(x)} I(X;Y)$$

9.17.



n个 $\quad Z_i \sim N(0,N)$

(a) $EX_1^2 = nP$ $\quad EX_i^2 = 0$ $\quad (i=2,3,\cdots,n)$

求 $\max_{f(x^n)} \frac{I(X^n;Y^n)}{n}$

$Y_i = Z_i,\ i=2,\cdots,n$

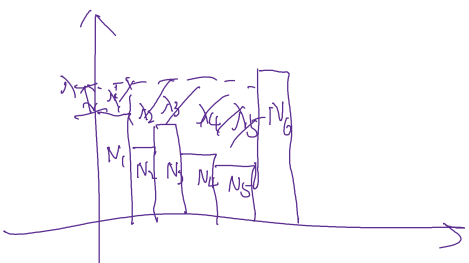
$H(Y_i) = H(Z_i)$

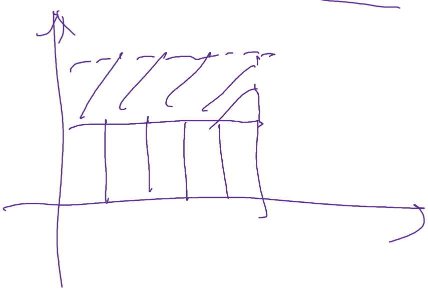
$$\begin{aligned} I(X^n;Y^n) &= H(Y^n) - H(Y^n|X^n) \\ &= H(Y^n) - H(Z^n+X^n|X^n) \\ &= H(Y^n) - H(Z^n|X^n) \\ &= H(Y^n) - H(Z^n) \\ &= H(Y_1,Y_2,\cdots,Y_n) - H(Z^n) \\ &\le \sum_{i=1}^{n} H(Y_i) - \cancel{H(Z^n)} \sum_{i=1}^{n} H(Z_i) \\ &= H(Y_1) - H(Z_1) \\ &\le \tfrac{1}{2}\log(1+\tfrac{nP}{N}) \end{aligned}$$

$$\frac{\frac{1}{2}\log(1+\frac{nP}{N})}{n}$$

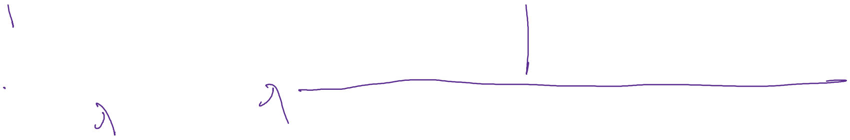
$$\frac{1}{2}\log(1+\frac{P}{N})$$

(b) 求 $\max_{f(x^n):\ E(\frac{1}{n}\sum_{i=1}^{n} x_i^2) \le P} \frac{1}{n} I(X^n;Y^n)$





9.19.



$N_1 \quad N_2 \quad N_3 \quad \cdots \quad N_k$

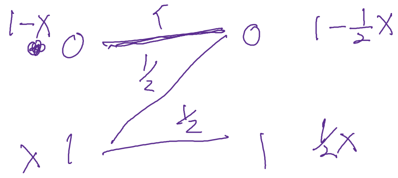
$$P = \sum_{i=1}^{m} (\lambda - N_i)^+ \qquad x^+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$C = \sum_{i=1}^{m} \frac{1}{2} \log\left(1 + \frac{(\lambda - N_i)^+}{N_i}\right)$$

9.4 $\quad \boxed{9.1 - 9.9.}$

$\boxed{9.21}$

7.8. $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$



$I(X;Y) \leq \boxed{H(X)}$

$x = P_r(X=1)$

$I(X;Y) = H(Y) - H(Y|X)$

$= H(\frac{x}{2}) - x$

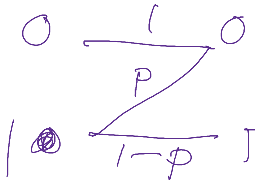
$H(Y|X)$

$= p(x=0)\, H(Y|X=0) = 0$

$+ p(x=1)\, H(Y|X=1) = x$

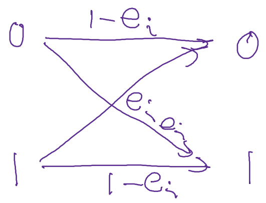
$$\frac{dI(X;Y)}{dx} = -\frac{1}{2}\log\frac{x}{2} - \frac{1}{2} + \frac{1}{2}\log(1-\frac{x}{2}) + \frac{1}{2} - 1$$

$$= \frac{1}{2}\log\frac{2-x}{4x} = 0 \Rightarrow x = 2/5$$



$C = H(\frac{1}{5}) - 2/5 \approx 0.322$

7.11



$p(\vec{y}, \vec{x}) = \prod_{i=1}^{n} p(y_i | x_i)$

$\max I(X^n; Y^n)$

$I(X^n; Y^n) = H(Y^n) - H(Y^n | X^n)$

$= H(Y^n) - \sum_{i=1}^{n} H(Y_i | Y_1, \cdots Y_{i-1}, X^n)$

$= H(Y^n) - \sum_{i=1}^{n} H(Y_i | X_i)$

X_i 独立 ~~i.i.d.~~ $\longrightarrow \leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i | X_i)$

$= \sum_{i=1}^{n} I(X_i; Y_i)$

$X_i \sim Bern(\frac{1}{2}) \longrightarrow \leq \sum_{i=1}^{n} (1 - h(P_i))$

7.12

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 3 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$

$P_1 + P_2 + P_3 = 1$

$p_1 \qquad p_2 \qquad p_3 \qquad\qquad (\frac{2}{3}p_1+\frac{1}{3}p_2,\ \frac{1}{3},\ \frac{1}{3}p_2+\frac{2}{3}p_3)$

$$I(X;Y) = H(Y) - \underline{H(Y|X)}$$
$$= H(\tfrac{2}{3}p_1+\tfrac{1}{3}p_2,\ \tfrac{1}{3},\ \tfrac{1}{3}p_2+\tfrac{2}{3}p_3) - \underline{(p_1+p_3)H(\tfrac{2}{3},\tfrac{1}{3}) - p_2\log 3}$$
$$= \underline{H(\tfrac{1}{3}+\tfrac{1}{3}(p_1-p_3),\ \tfrac{1}{3},\ \tfrac{1}{3}-\tfrac{1}{3}(p_1-p_3))} - \underline{(p_1+p_3)}H(\tfrac{2}{3},\tfrac{1}{3}) - (1-\underline{p_1-p_3})\log 3$$

$$H(\tfrac{1}{3}+\tfrac{1}{3}(p_1-p_3),\ \tfrac{1}{3},\ \tfrac{1}{3}-\tfrac{1}{3}(p_1-p_3)) \le H(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) = \log 3$$

等号 $\Leftrightarrow$ $\underline{p_1=p_3}$

$$-(p_1+p_3)H(\tfrac{2}{3},\tfrac{1}{3}) - \log 3 + (p_1+p_3)\log 3$$
$$= (\underline{\log 3 - H(\tfrac{2}{3},\tfrac{1}{3})})(p_1+p_3) - \log 3$$
$$= \tfrac{2}{3}(p_1+p_3) - \log 3$$
$$\le \tfrac{2}{3} - \log 3$$

等号 $\Leftrightarrow$ $\underline{p_1+p_3=1}$

$$\log 3 - H(\tfrac{2}{3},\tfrac{1}{3})$$
$$= \log 3 + \tfrac{2}{3}\log\tfrac{2}{3} + \tfrac{1}{3}\log\tfrac{1}{3}$$
$$= \underline{\tfrac{2}{3}}$$

当 $p_1=p_3=\frac{1}{2}$ 时

$I(X;Y)$取到最大值

$\cancel{I(X)}$ $\log 3 + (\frac{2}{3} - \log 3)$

$= \frac{2}{3}$

$\boxed{9.21 \qquad 7.28 \qquad 7.36}$