Lecture 5: Differential Equations

September 27, 2023

- Consider the first-order differential equation $\dot{x} = f(x)$, where we assume that f is Lipschitz-continuous.
- Consider the set of zeros of f, which are the constant solutions (equilibria).
- The set of zeros is a closed set because f is continuous.
- Therefore its complement is open and can be written as a disjoint union of open intervals.

Lemma 1.1

Consider a Lipschitz-continuous function f and suppose $f \neq 0$ on (a, b) and $f(a) = f(b) = 0$. Then, for any initial condition $x_0 \in I$, the corresponding solution of $\dot{x} = f(x)$ is monotone. It is increasing (and asymptotic to b) if $f > 0$ on I, decreasing (and asymptotic to a) otherwise.

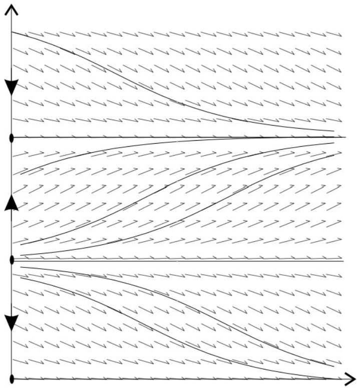
Proof.

- Suppose $f(x_0) > 0$ to be definite (the other case works the same way).
- It is easy to see that the solution increases so long as it is in (a, b).
- The point is to show that it can't leave that interval.
- Since $\dot{x}(0) = f(x(0)) = f(x_0) > 0$, the solution initially increases.
- If it ever becomes decreasing, then we must have a maximum $x(t_0) = c$ at that time, which implies that $f(c) = 0$ and therefore $c = b$.
- We need to check that this never happens, that is, $x(t) \neq b$ for all time.
- For this there are two ways, the honest one and the easy one.

Proof.

- We begin with the honest one.
- We can write the solution of a differential equation $\dot{x} = f(x)$ as $x(t) = x(0) + \int_0^t f(x(s))ds$.
- For our problem write $dx/dt = f(x)$ and by the Inverse-Function Theorem $dt/dx = 1/f(x)$, so in integral form $t(x) = \int_{x_0}^{x}(1/f(s))ds$.
- Since f is Lipschitz-continuous, we have $f(s) = f(s) - f(b) \leq C(b - s)$ for some constant C.
- Therefore,
$$t(x) = \int_{x_0}^{x} \frac{1}{f(s)} ds \geq \int_{x_0}^{x} \frac{1}{C(b-s)} ds.$$
- If $x = b$, then this integral diverges, that is, $t(x) = \infty$. This shows that $x(t) < b$ for all (finite) t and furthermore that $x(t) \to b$ as $t \to \infty$.

- The easy way to do the last portion is to use existence and uniqueness of solutions to differential equations.
- Because $\tilde{x}(t) = b$ for all $t \in \mathbb{R}$ is a solution that takes the value b at sometime (any time), any solution that ever reaches b is of the form $\tilde{x}(t - t_0) = b$.
- Since we did not start from b, our solution is not this one, so it never reaches b.



Proposition 1.2

When $f(x_0) = 0$ and $f'(x_0) < 0$, then x_0 is an attracting fixed point of $\dot{x} = f(x)$: Every nearby orbit is positively asymptotic to x_0. Likewise, fixed points x_0 with $f'(x_0) > 0$ are repelling: Nearby points move away from x_0.

Proof.

If $f'(x_0) < 0$, then $f(x) < 0$ for $x > x_0$ nearby, so such points move toward x_0, and vice versa for $x < x_0$. Thus every nearby orbit is positively asymptotic to x_0. □

- The simplest model of this nature is one involving exponential growth.
- Suppose that at any given time the rate of births and deaths is a constant percentage of the total population at that time.
- That is, there is a constant k such that if the real variable x denotes the population then $\dot{x} = kx$ or $(d/dt)x = kx$.

Lemma 2.1

The solution of $\dot{x} = kx$ is $x(t) = x(0)e^{kt}$.

- For larger populations the limited amount of food and possibly other resources play a role.
- Thus there should be a saturation population that does not grow any more; and if the population were to start out at a higher number, it should shrink to the saturation level.
- Thus, in a manner of speaking, k should be a function of x that is zero at a (positive) saturation value L of x (no growth) and negative for larger values of x (shrinking population).
- If we take a linear function $k = a(Lx)$ with $a > 0$ to do this, then we get the differential equation

$$\frac{d}{dt}x = ax(L - x).$$

Lemma 2.2

The solution of $\dot{x} = ax(L - x)$ is

$$x(t) = \frac{Lx(0)}{x(0) + (L - x(0))e^{-Lat}}.$$

Proof.

We separate variables, that is, bring all x's to one side:

$$a = \frac{dx/dt}{x(L-x)}.$$

Integrating over t gives

$$at+C = \int \frac{1}{x(L-x)}\frac{dx}{dt}dt = \int \frac{1}{x(L-x)}dx = \int \frac{1}{Lx}dx + \int \frac{1}{L(L-x)}dx$$

using partial fractions. Thus

$$at + C = \frac{\log|x|}{L} - \frac{\log|L-x|}{L} = \frac{1}{L}\log|\frac{x}{L-x}|.$$

Proof.

Taking $t = 0$ shows that

$$CL = \log|\frac{x(0)}{L - x(0)}|, \text{ hence } e^{-CL} = |\frac{L}{x(0)} - 1|.$$

changing sign and exponentiating gives

$$e^{-Lat}|\frac{L - x(0)}{x(0)}| = e^{-L(at+C)} = |\frac{L}{x} - 1| = |\frac{L - x(t)}{x(t)}| = |\frac{L}{x(t)} - 1|.$$

The quantities in absolute value signs turn out to always agree in sign, so we can drop the absolute values. This gives

$$x(t) = \frac{Lx(0)}{x(0) + (L - x(0))e^{-Lat}}.$$

□

- We develop the asymptotic behavior of the solutions to this differential equation.
- For $x(0) = L$ we get the expected constant solution $x(t) = L$. When $t \to +\infty$, the exponential term goes to zero; hence $x(t) \to L$ for any positive initial condition.
- If $x(0) < L$, then as $t \to -\infty$ the exponential term diverges and $x(t) \to 0$. For $x(0) > L$ and $x(0) < 0$ (the latter is biologically meaningless) the denominator is zero (the solution has a singularity) for

$$t = \frac{\log(1 - |L/x(0)|)}{La},$$

which is negative for $x(0) > L$ and positive for $x(0) < 0$.

- Therefore the asymptotic behavior for positive time is simple.
- If the initial population is zero, then it remains zero forever.
- If the initial population is positive but below saturation (that is, less than L), then the population increases and in the long run creeps up to the saturation population.
- The increase is most rapid when the population is $L/2$ because $x(L-x)$ is maximal at $L/2$.
- Initial population larger than L shrink to L asymptotically.

- We now produce the less obvious continuous-time analog of attracting fixed points for maps.
- We use properties of flows.
- The obvious analogs are attracting fixed points for flows such as the saturation population L in the previous example.
- The second analog cannot be found on the line. It is an attracting periodic orbit (periodic solution) for a differential equation in the plane or in higher dimension.

Lemma 3.1

If p is T-periodic and not fixed for $\dot{x} = f(x)$, then 1 is an eigenvalue of $D\phi_p^T$.

Proof.

$f(p) = f(\phi^T(p)) = (d/ds)\phi^s(p)|_{s=T} = (d/ds)\phi^T \circ \phi^s(p)|_{s=0} = D\phi_p^T f(p)$.
Thus, $f(p)$ is an eigenvector for $D\phi_p^T$ with eigenvalue 1. □

Definition 3.2

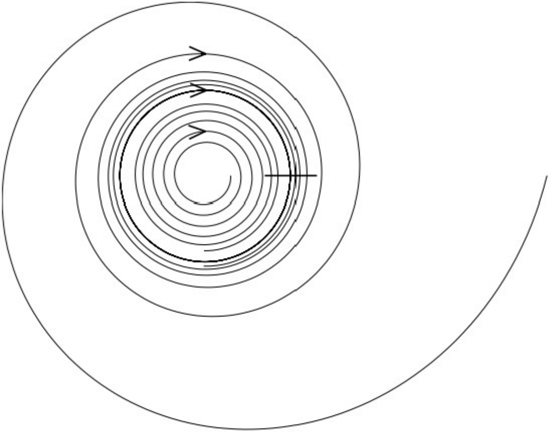
If p is a T-periodic point and the eigenvalues of $D\phi_p^T$ are $\lambda_1, \cdots, \lambda_{n-1}, 1$ (not necessarily distinct), then $\lambda_1, \cdots, \lambda_{n-1}$ are called the *eigenvalues* at p.

Remark 3.3

These eigenvalues depend only on the orbit: If $q = \phi^s(p)$, then $\phi^T \circ \phi^s = \phi^s \circ \phi^T$ implies $D\phi_q^T D\phi_p^s = D\phi_p^s D\phi_p^T$, that is, the linear maps $D\phi_q^T$ and $D\phi_p^T$ are conjugate via $D\phi_p^s$; hence the eigenvalues at p and q coincide.

Proposition 3.4

If p is a periodic point with all eigenvalues of absolute value less than 1, then the orbit $\mathcal{O}(p)$ of p is a limit cycle, that is, it has a neighborhood whose every point is positive asymptotic to $\mathcal{O}(p)$.



Proof.

- We construct a map that reflects the dynamics and the eigenvalue information.
- To that end consider the flow direction at p and pick a small piece from its orthogonal subspace.
- This is a little disk S containing p such that the orbit of p crosses it.
- We need to use continuity up to time $1.1 \cdot T$, say several times, which can be stated thus:

Lemma 3.5

Given $\epsilon > 0$, there is a $\delta > 0$ such that any point within δ of $\mathcal{O}(p)$ will remain within ϵ of $\mathcal{O}(p)$ for time $1.1 \cdot T$.

Proof.

- Taking ϵ such that S contains an ϵ-disk around p we find that, whenever $q \in S$ is sufficiently close to p, its orbit again intersects S after time less than $1.1 \cdot T$.
- This means that on a neighborhood of p in S there is a well-defined return map F_p^S.
- By smoothness and the Implicit-Function Theorem F_p^S is smooth.

Proposition 3.6

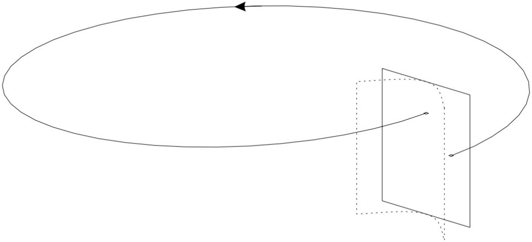
The eigenvalues at p coincide with those of DF_p^S.

Proof.

- If we denote the projection to S parallel to $f(p)$ by $\pi : \mathbb{R}^n \to S$ then the differential of $F^S(x) = \phi^{t_x}(x)|_S$ as a map into S is

$$DF_p^S = \pi(D\phi_p^{t_p}|_S + \dot{\phi}^{t_p}(p)Dt_p|_S).$$

- Applying π to $\dot{\phi}^{t_p}(p)Dt_p|_S = f(\phi^{t_p}(p))Dt_p|_S = f(p)Dt_p|_S$ gives zero, so $DF_p^S = \pi D\phi_p^{t_p}|_S =: A$. But, on the other hand, extending a basis of S to one of $\mathbb{R}^n$ by adding $f(p)$ gives the coordinate representation $D\phi_p^{t_p} = \begin{pmatrix} A & 0 \\ * & 1 \end{pmatrix}$. □



Proof.

- This means that p is an attracting fixed point of F^S with a neighborhood $U \subset S$ of attraction.
- Every point close enough to the orbit of p will encounter U and from then on do so at intervals less than $1.1 \cdot T$.
- The resulting return points converge to p.
- So the entire positive semiorbit of q then converges to p. □