Lecture 8: Linear Differential Equations in the Plane, Linear Maps and Differential Equations in Higher Dimension

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- The appearance of invariant curves in the examples in the last lecture is not accidental.
- The linear maps we described above arise from solutions of closely related differential equations, whose solutions interpolate iterates of the maps above. These are of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Or, more explicitly,

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y \\ \dot{y} &= a_{21}x + a_{22}y. \end{aligned}$$

Node

- The case of a right-hand side with two distinct positive (real) eigenvalues is represented by the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \log \lambda & 0 \\ 0 & \log \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  where $\lambda, \mu > 1$.

- Its solutions are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{t \log \lambda} \\ y(0)e^{t \log \mu} \end{pmatrix} = \begin{pmatrix} x(0)\lambda^t \\ y(0)\mu^t \end{pmatrix} = \begin{pmatrix} \lambda^t & 0 \\ 0 & \mu^t \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

- For $t = 1$ we get the map

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

  and therefore the solution curves parameterize the invariant curves that we found before.

- As in the corresponding discrete-time case, this orbit picture is called a *node*.

- Corresponding to the linear map

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda > 1$ consider the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \log\lambda & 1 \\ 0 & \log\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Its solutions are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{t\log\lambda} + y(0)te^{t\log\lambda} \\ y(0)e^{t\log\lambda} \end{pmatrix} = \begin{pmatrix} \lambda^t & t\lambda^t \\ 0 & \lambda^t \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \lambda^t \begin{pmatrix} x(0) + ty(0) \\ y(0) \end{pmatrix}.$$

- Here the maps of the space with parameter t are given by the matrices

$$\begin{pmatrix} \lambda^t & t\lambda^t \\ 0 & \lambda^t \end{pmatrix}$$

and for $t = n$ this gives the action of

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n.$$

- Linear maps with eigenvalues $\rho e^{\pm i\theta}$ arise from the linear differential equation

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\log\rho & \theta\\ -\theta & \log\rho\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

  with solutions

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \rho^t\begin{pmatrix}x(0)\cos\theta t + y(0)\sin\theta t\\ y(0)\cos\theta t - x(0)\sin\theta t\end{pmatrix} = \rho^t\begin{pmatrix}\cos\theta t & \sin\theta t\\ -\sin\theta t & \cos\theta t\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}.$$

- Here the solution family gives maps with matrices

$$\rho^t\begin{pmatrix}\cos\theta t & \sin\theta t\\ -\sin\theta t & \cos\theta t\end{pmatrix}$$

  the expected rotation with scaling, and these solutions parameterize the invariant spirals.
- Following the discrete-time case, this picture is called a *focus*.
- The exceptional case is $|\rho| = 1$, where the solutions are pure rotations and every circle $r =$ const. is a periodic orbit and is called a *center*.

- The continuous-time picture of a *saddle* is obtained from

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}.$$

where A has one positive and one negative eigenvalue.

- That is, by considering the differential equation

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\log\lambda & 0\\ 0 & \log\mu\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

- Its solution are given by

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}x(0)e^{t\log\lambda}\\ y(0)e^{t\log\mu}\end{pmatrix} = \begin{pmatrix}x(0)\lambda^t\\ y(0)\mu^t\end{pmatrix} = \begin{pmatrix}\lambda^t & 0\\ 0 & \mu^t\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}.$$

- For $t = 1$ we get the map

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},$$

and therefore the solution curves parameterize the invariant curves that we found before.

- As in the corresponding discrete-time case, this orbit picture is called a *saddle*.

The matrix exponential

- The connection between linear maps on the plane and two-dimensional linear differential equations with constant coefficients can be made explicit by recalling that the solution to the differential equation $\dot{x} = ax$ with $x(0) = x_0$ is given by $x(t) = e^{at}x_0$.
- Analogously, the solution of $\dot{x} = AX$ with $x \in \mathbb{R}^n$ and an $n \times n$ matrix A is

$$x(t) = e^{At}x(0), \text{ where } e^{At} := \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}.$$

- The series converges absolutely because every entry of A^i is bounded in absolute value by $\|A^n\| \leq \|A\|^n$.

For example, if

$$A = \begin{pmatrix} \log\lambda & 0 \\ 0 & \log\mu \end{pmatrix},$$

then

$$e^{At} = \begin{pmatrix} \lambda^t & 0 \\ 0 & \mu^t \end{pmatrix}$$

because

$$\begin{aligned} \sum_{i=0}^{\infty} \begin{pmatrix} (\log\lambda)^i & 0 \\ 0 & (\log\mu)^i \end{pmatrix} \frac{t^i}{i!} &= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(\log\lambda)^i t^i}{i!} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{(\log\mu)^i t^i}{i!} \end{pmatrix} \\ &= \begin{pmatrix} e^{t\log\lambda} & 0 \\ 0 & e^{t\log\mu} \end{pmatrix}. \end{aligned}$$

It is only slightly less straightforward to check that

$$e^{\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} t} = \begin{pmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{pmatrix}$$

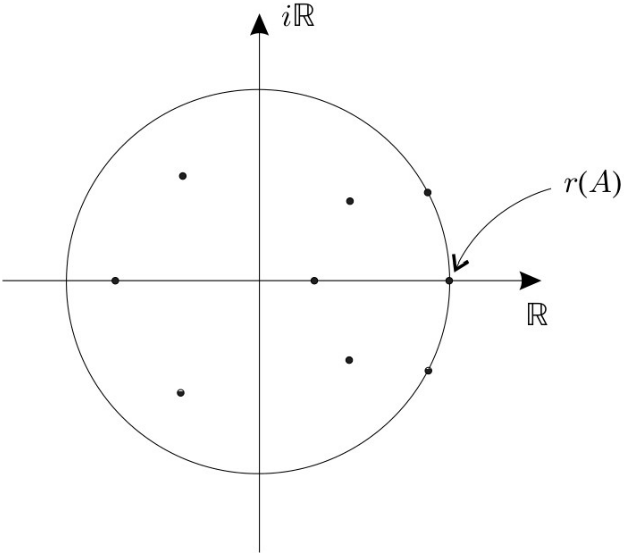
and

$$e^{\begin{pmatrix} \log \rho & \theta \\ -\theta & 0 \log \rho \end{pmatrix} t} = \rho^t \begin{pmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{pmatrix}.$$

- In the previous situation the (linear) time-1-map is e^A.
- For an initial condition $x(0)$ it gives the solution $x(1)$ at time 1.
- To get $x(n)$, we iterate this map n times.
- This works because the differential equations do not involve the time parameter, or $e^{Ai} = (e^A)^i$.
- If $\dot{x} = A(t)x$, where $A(t+1) = A(t)$ and M is such that $x(1) = Mx(0)$ for any solution $x(\cdot)$, then we find $x(2)$ from $x(0)$ by solving the differential equation $\dot{x} = A(t+1)x = A(t)x$ with initial condition $x(1)$ for time 1.
- Hence $x(2) = M^2x(0)$ and inductively $x(i) = M^ix(0)$.
- The same reasoning and conclusion apply to differential equation $\dot{x} = f(x,t)$ with $f(x,t+1) = f(x,t)$ for $x \in \mathbb{R}^n$.

Definition 2.1

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. We call the set of eigenvalues the *spectrum* of A and denote it by sp A. We denote the maximal absolute value of an eigenvalue of A by $r(A)$ and call it the *spectral radius* of A.


$i\mathbb{R}$
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Lemma 2.2

For an $n \times n$ matrix A denote its entries by a_{ij} and define the norm $|A| := \max_{ij} |a_{ij}|$. Then $|A| \leq \|A\| \leq \sqrt{n}|A|$.

Proof.

We have that

$$\|Av\| = \sqrt{\sum_{i=1}^{n}(\sum_{i=1}^{n} a_{ij}v_j)^2} \leq |A|\sqrt{\sum_{i=1}^{n}(\sum_{i=1}^{n} v_j)^2} = \sqrt{n}|A|\|v\|,$$

and conversely

$$|a_{ij}| = \langle e_i, Ae_j \rangle \leq \|A\|.$$

□

Proposition 2.3

For every $\delta > 0$ there is a norm in $\mathbb{R}^n$ such that $\|A\| < r(A) + \delta$.

Lemma 2.4

Consider $\mathbb{R}^n$ with any norm $\|\cdot\|$ and a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. If $C, \lambda > 0$ are constants such that $\|A^n\| \leq C\lambda^n$ for all $n \in \mathbb{N}$, and if $\mu > \lambda$, then there is a norm $\|\cdot\|'$ on $\mathbb{R}^n$ with respect to which $\|A\| < \mu$.

Proof.

If $n \in \mathbb{N}$ is such that $C(\lambda/\mu)^n < a$, then $\|v\|' := \sum_{i=0}^{n-1} \|A^i v\|/\mu^i$ defines a nrom with

$$\|Av\|' = \sum_{i=1}^{n} \|A^i v\|/\mu^{i-1} = \mu(\|v\|' + \frac{A^n v\|}{\mu^n} - \|v\|)$$
$$\leq \mu\|Av\|' - (1 - C\frac{\lambda^n}{\mu^n})\|v\| \leq \mu\|Av\|'.$$

□

Proof of Proposition 2.3

Proof.

By the lemma it suffices to show that there is a coordinate change and a norm for which $\|A^n\| \leq C(r(A) + (\delta/2))^n$ for all $n \in \mathbb{N}$.

For each real eigenvalue λ (with multiplicity k) consider the *generalized eigenspace* or *root space* $E_\lambda := \{v | (A - \lambda \mathrm{Id}^k v = 0\}$. With a little linear algebra one can see that $\dim(E_\lambda) = k$. Thus, these spaces generate the whole space. Alternatively, this follows from the Jordan normal form.

Proof.

On E_λ the binomial formula gives

$$A^n = (\lambda \mathsf{Id} + \Delta)^n = \sum_{l=0}^{k-1} \binom{n}{l} \lambda^{-l} \Delta^l = \lambda^n \sum_{l=0}^{k-1} \binom{n}{l} \lambda^{-l} \Delta^l.$$

The entries of $\Delta_n := \sum_{l=0}^{k-1} \binom{n}{l} \lambda^{-l} \Delta^l$ are polynomials in n. If $\delta > 0$, then $p(n)|\lambda|^n/(|\lambda| + \delta)^n \to 0$; so there is $C > 0$ such that

$$\|A^n\|/|\lambda|^n \leq p(n) \leq C(|\lambda| + (\delta/2))^n/|\lambda|^n$$

for all n.

Proof.

One can analyze complex eigenvalues separately, or one can look at A as a linear map of $\mathbb{C}^n$ by allowing complex numbers as components of vectors. Then the preceding analysis applies to the root spaces for complex eigenvalues as well.

To deduce the complete result from the result for root spaces we use a norm of the desired kind on each of these root spaces. If we write vectors v as $(v_1, \cdots, v_l)$ with every v_l in a different root space with norm $\|\cdot\|_l$, then we can define the desired norm by $\|v\| := \sum_{i=1}^{l} \|v_l\|_l$. □

Corollary 2.5

If $r(A) < 1$, then A is eventually contracting. In particular, the positive iterates of every point converge to the origin with exponential speed. If, in addition, A is an invertible map, that is, if zero is not an eigenvalue for A, then negative iterates of every point go to infinity exponentially.

Lemma 2.6

Let f be a continuous differentiable map with a fixed point x_0 where $r(Df_{x_0}) < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is eventually contracting on U.

- It remains to understand the mixed situations with only some eigenvalues inside or outside the unit circle.
- Analogously to the two-dimensional situation, it is possible to decompose $\mathbb{R}^n$ into subspaces that are contracting, expanding, or neutral, except that now all three possibilities can coexist.
- As in the two-dimensional case, these subspaces correspond to sets of eigenvalues inside, outside, and on the unit circle, respectively.
- But, similarly to the case of only one real eigenvalue in $\mathbb{R}^2$, it does not quite suffice to consider eigenspaces only.
- Instead, we have to consider the generalized eigenspaces or root spaces of A introduced in the preceding proof for the case of real eigenvalues (while this much generality is desirable, the reader may choose to assume diagonalizability to make the argument more transparent).

- For a pair of complex conjugate eigenvalues $\lambda, \bar{\lambda}$ we let $E_{\lambda,\bar{\lambda}}$ be the intersection of $\mathbb{R}^n$ with the sum of root spaces corresponding to E_λ and $E_{\bar{\lambda}}$ for the complexification of A (that is, the extension to the complex space $\mathbb{C}^n$)
- For brevity we call $E_{\lambda,\bar{\lambda}}$ a root space, too.
- Let

$$E^- = E^-(A) = \bigoplus_{-1<\lambda<1} E_\lambda \oplus \bigoplus_{|\lambda|<1} E_{\lambda,\bar{\lambda}}$$

be the space jointly spanned by all root spaces for eigenvalues inside the unit circle and similarly

$$E^+ = E^+(A) = \bigoplus_{|\lambda|>1} E_\lambda \oplus \bigoplus_{|\lambda|>1} E_{\lambda,\bar{\lambda}}$$

- If the map A invertible, then $E^+(A) = E^-(A^{-1})$.
- Finally, let

$$E^0 = E^0(A) = E_1 \oplus E_{-1} \bigoplus_{|\lambda|=1} E_{\lambda,\bar{\lambda}}.$$

- The spaces $E^-,\ E^+,\ E^0$ are obviously invariant with respect to A and $\mathbb{R}^n = E^- \oplus E^+ \oplus E^0$.

Corollary 2.7

The restriction $A|_{E^-(A)}$ of a linear map A to the space $E^-(A)$ is eventually contracting. If A is invertible., then in addition $A^{-1}|_{E^+(A)}$ is eventually contracting. Furthermore, for any $\delta > 0$ there is a norm with respect to which $\|A|_{E^-(A)}\| \leq r(A|_{E^-(a)}) + \delta$ and $\|A^{-1}|_{E^+(A)}\| \leq r(A^{-1}|_{E^+(A)}) + \delta$.

To obtain this Lyapunov norm apply Proposition 2.3 on $E^-(A)$ and $E^+(A)$ separately to get norms $\|\cdot\|_-$ and $\|\cdot\|_+$, respectively, on these such spaces. Then define a norm for points $x = (x_-, x_0, x_+)$ by $\|(x_-, x_0, x_+)\| := \|x_-\| + \|x_0\| + \|x_+\|_+$.

Definition 2.8

The space $E^{-}(A)$ above is called the *contracting subspace* and the space $E^{+}(A)$, the *expanding subspace*. We say that A is *hyperbolic* if $E^{0} = \{0\}$ or, equivalently, if $\mathbb{R}^n = E^{+} \oplus E^{-}$.