Lecture 9: Rotations of the Circle

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- We have seen two different convenient ways to represent the circle that allow us to write various formulas in a nice fashion.
- **Multiplicative notation:** the circle represented as the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} | |z| = 1\} = \{e^{2\pi i \phi} | \phi \in \mathbb{R}\}$$

  All algebraic operations make sense as operations over complex numbers.
- **Additive notation:** $S^1 = \mathbb{R}/\mathbb{Z}$ consists of the real numbers with integer translates identified. We can use addition and subtraction just as the usual operations over real numbers, but we have to keep in mind that all equalities make sense up to an integer. It is customary to add "$(\mathrm{mod} 1)$" to such equalities. Thus, the expression $a = b \mod 1$, where a and b are real numbers, means that $a - b$ is an integer.
- The logarithm map $e^{2\pi i \phi} \mapsto \phi$ establishes an isomorphism between these representations.

- Let us measure the length of arcs on the circle by the parameter ϕ; that is, the length of the whole circle is equal to one.
- Let $l(\Delta)$ denote the length of the arc Δ measured in such a way.
- We can similarly define a distance on the set

$$X = \mathbb{R}/\mathbb{Z} := \{[x] | x \in \mathbb{R}\}$$

of equivalence classes by setting

$$d(x, y) := \min\{|b - a| a \in x, b \in y\}$$

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- We use the symbol R_α to denote the notation by the angle $2\pi\alpha$.
- In multiplicative notation

$$R_\alpha(z) = z_0 z \text{ with } z_0 = e^{2\pi i\alpha}.$$

- In additive notation we have

$$R_\alpha = x + \alpha \pmod 1.$$

- The iterates of the rotation are correspondingly

$$R_\alpha^n(z) = R_{n\alpha}(z) = z_0^n z$$

in multiplicative notation and

$$R_\alpha^n(x) = x + n\alpha \pmod 1.$$

in additive notation.

- A crucial distinction in the dynamics of rotations appears between the cases of the rotation parameter α being rational and irrational.
- In the former case, write $\alpha = p/q$, where p, q are relatively prime integers. Then $R_\alpha^q(x) = x$ for all x, so R_α^q is the identity map and after q iterates the transformation simply repeats itself.
- Thus the total orbit of any point is a finite set and all orbits are q-periodic.
- The case of irrational α is much more interesting.

Theorem 2.1

If $\alpha \notin \mathbb{Q}$, then every positive semiorbit of R_α is dense.

Remark 2.2

Since the negative semiorbit of R_α is the positive semiorbit of $R_{-\alpha}$, we also have the density of negative semiorbits.

Proof.

Suppose $x, z \in S^1$. To show that z is in the closure of the positive semiorbit of x, let $\epsilon > 0$. The positive semiorbit of x is infinite and no set of $k \geq \lfloor 1/\epsilon \rfloor + 1$ points has pairwise distances all exceeding ϵ. Thus there are $l.k \in \mathbb{N}$ such that $l < k \leq \lfloor 1/\epsilon \rfloor$ and $d(R_\alpha^k(x), R_\alpha^l(x)) < \epsilon$. Then $d(R_\alpha^{k-l}(x), x) < \epsilon$ because R_α^{-l} preserves distances. By the way, this latter distance is independent of x because, if $y \in S^1$, then $y = R_{y-x}(x)$ and

$$\begin{aligned} d(R_\alpha^{k-l}(y), y) &= d(R_\alpha^{k-l}(R_{y-x}(x)), R_{y-x}(x)) = d(R_{(k-l)\alpha+y-x}(x), R_{y-x}(x)) \\ &= d(R_{y-x}(R_\alpha^{k-l}(x)), R_{y-x}(x)) = d(R_\alpha^{k-l}(x), x); \end{aligned}$$

so k and l can be chosen independently of x.

Take $\theta \in [-1/2, 1/2]$ such that $\theta = (k-l)\alpha \mod 1$. Then $\rho := |\theta| < \epsilon$ and $R_\alpha^{k-l} = R_\theta$. Let $N = \lfloor 1/\rho \rfloor + 1$ (independently of x). Then the subset $\{R_{i\theta} | i = 0, 1, \cdots, N\}$ of the positive semiorbit of x divides the circle into intervals of length less than $\rho < \epsilon$, so there is an $n \leq N(k-1)$ such that $d(R_\alpha^n(x), z) < \epsilon$. □

Alternate proof.

Let $A \subset S^1$ be an invariant closed set. The complement $S^1 \backslash A$ is a nonempty open invariant set that consists of disjoint intervals.
Let I be the longest of those intervals (or one of the longest, if there are several of the same length). Since rotation preserves the length of any interval, the iterates $R_\alpha^n(I)$ do not overlap. Otherwise $S^1 \backslash A$ would contain an interval longer than I. Since α is irrational, no iterates of I can coincide, because then an endpoint x of an iterate of I would come back to itself and we would have $x + k\alpha = x \mod 1$ with $k\alpha = l$ an integer and $a\alpha = l/k$ a rational number.

Thus the intervals $R_\alpha^n(I)$ are all of equal length and disjoint, but this is impossible because the circle has finite length and the sum of lengths of disjoint intervals cannot exceed the length of the circle. □

Definition 2.3

A homemorphism $f : X \to X$ is said to be *transitive* if there exists a point $x \in X$ such that its orbit $\mathcal{O}_f(x) = (f^n(x))_{n\in\mathbb{Z}}$ is dense in X. Equivalently, every f-invariant open invariant set is dense. A noninvertible map f is said to be *transitive* if there exists a point $x \in X$ such that its (positive) orbit $\mathcal{O}_f^+(x) := (f^n(x))_{n\in\mathbb{N}_0}$ is dense in X.

Definition 2.4

A map $f : X \to X$ is called *topologically transitive* if given nonempty open sets $U, V \subset X$, there exists an $n \in \mathbb{N}$ such that $f^{-n}U \cap V \neq \emptyset$.

Definition 2.5

A homeomorphism $f : X \to X$ is said to be *minimal* if the orbit of every point $x \in X$ is dense in X or, equivalently, if f has no proper closed invariant subsets or, equivalently, if it is the orbit closure of any of its points.

- It may be interesting to get a good picture of how an orbit fills the circle densely.
- We do this in a specific example by following the orbit of 0 under a rotation R_α, where we take

$$\alpha = \frac{1}{3 + \frac{1}{5+\frac{1}{c}}}$$

for some $c > 1$. Note that $\alpha \in \mathbb{Q}$ if and only if $c \in \mathbb{Q}$.

- Since $1/4 < \alpha < 1/3$ and hence $3\alpha < 1 < 4\alpha$, the first time the orbit returns more closely to 0 than ever before is after three steps.
- The first three points $\alpha, 2\alpha$ and 3α, are evenly spaced, and since $4\alpha > 1$, 3α is closer to an integer than the previous points.
- The precise distance is

$$\delta := 1 - 3\alpha = 1 - \frac{3}{3 + \frac{1}{5+\frac{1}{c}}} = \frac{\frac{1}{5+\frac{1}{c}}}{3 + \frac{1}{5+\frac{1}{c}}} = \frac{1}{16 + \frac{3}{c}}.$$

- To find the next time of closest return we start from the fourth step, using $4\alpha = \alpha - \delta \bmod 1$.
- So three α-steps take us from α to $\alpha - \delta$.
- How many of these 3α-steps does it take to get the next closest approach?
- As before, it should be about α/δ, and the desired number n must satisfy $n\delta < \alpha < (n+1)\delta$.
- Indeed, $n = 5$ works:

$$5\delta = \frac{5}{15 + (1 + \frac{3}{c})} = \frac{1}{3 + (\frac{1}{5} + \frac{3}{5c})} < \frac{1}{3 + \frac{1}{5}} < \frac{1}{3 + \frac{1}{5 + \frac{1}{c}}} = \alpha,$$

and

$$6\delta = \frac{6}{16 + \frac{3}{c}} > \frac{6}{18} = \frac{1}{3} > \alpha.$$

- These five 3α-steps evenly fill the interval $(0, \alpha)$ and simultaneously its three image intervals.
- When this next closest return is reached, the orbit segment is a δ-dense subset of the circle spaces evenly (except for the smaller interval of the new closet return).
- The next closest return after this is determined by c, and it is safe to guess that it will happen after about c-steps.
- If c were about a billion, this would mean that it takes about a billion 5δ-steps until the next closet return, which is some 15 billion iterations of R_α.
- In particular, the first 7 billion iterations are guaranteed to leave gaps of $\delta/2 > 1/35$. So large entries in this continued fraction form of α are not a good thing for filling the circle well.

- Fix an arc $\Delta \subset S^1$, and for $x \in S^1$ and $n \in \mathbb{N}$ let
$$F_\Delta(x,n) := \text{card}\{k \in \mathbb{Z} | 0 \le k < n,\ R_\alpha^k(x) \in \Delta\}.$$
- This function is nondecreasing in n for fixed x and Δ.
- Since the positive semiorbit of any point is dense, there are arbitrarily large positive iterates of x that belong to Δ.
- Hence
$$F_\Delta(x,n) \to \infty,\ \text{as } n \to \infty.$$
- The natural measure of how often these visits happen is the *relative frequency* of visits:
$$\frac{F_{\Delta(x,n)}}{n}.$$

Proposition 3.1

Suppose α is irrational and consider the rotation R_α. Let Δ, Δ' be arcs such that $l(\Delta) < l(\Delta')$. Then there exists an $N_0 \in \mathbb{N}$ such that, if $x \in S^1$, $N \geq N_0$, and $n \in \mathbb{N}$, then

$$F_{\Delta'}(x, n + N) \geq F_\Delta(x, n).$$

- It is convenient to always take arcs closed on the left and open on the right.
- For such arcs we have the following *additivity* property: If the right end of Δ_1 coincides with the left end of Δ_2, then $\Delta_1 \cap \Delta_2 = \emptyset$, $\Delta_1 \cup \Delta_2$ is an arc and

$$F_{\Delta_1}(x,n) + F_{\Delta_2}(x,n) = F_{\Delta_1 \cup \Delta_2}(x,n).$$

- It is also convenient to define

$$F_A(x,n) := \text{Card}\{k \in \mathbb{Z} | 0 \leq k < n,\ R_\alpha^k(x) \in A\}$$

for any set A that is union of disjoint arcs.

- One can consider the upper limits of relative frequencies:

$$\overline{f}_x(A) := \limsup_{n\to\infty} \frac{F_A(x,n)}{n}.$$

- These quantities are obviously subadditive:

$$\overline{f}_x(A_1 \cup A_2) = \overline{f}_x(A_1) + \overline{f}_x(A_2).$$

Corollary 3.2

If $l(\Delta) < l(\Delta')$, then $\overline{f}_x(\Delta) \leq \overline{f}_x(\Delta')$.

Similarly we introduce the lower asymptotic frequencies:

$$\underline{f}_x(A) := \liminf_{n\to\infty} \frac{F_A(x,n)}{n}.$$

Obviously, for any set A we have $F_A(x,n) = n - F_{A^c}(x,n)$, where A^c denotes the complement $S^1 \backslash A$ of A and hence

$$\overline{f}_x(A) := \limsup_{n\to\infty} \frac{F_A(x,n)}{n} = 1 - \liminf_{n\to\infty} \frac{F_{A^c}(x,n)}{n} = 1 - \underline{f}_x(A^c).$$

Proposition 3.3

For any arc $\Delta \subset S^1$ and any $x \in S^1$,

$$f(\Delta) := \lim_{n\to\infty} \frac{F_\Delta(x,n)}{n} = l(\Delta),$$

and the limit is uniform in x.

Lemma 3.4

If $l(\Delta) = 1/k$, then $\overline{f}_x(\Delta) \le 1/(k-1)$.

Proof.

Consider $k-1$ disjoint arcs $\Delta_1, \Delta_2, \cdots, \Delta_{k-1}$ of length $1/(k-1)$ each. For $1 \le i < k$, Proposition 3.1 gives natural numbers N_i such that, if $x \in S^1$, then

$$F_{\Delta_i}(x, n + N_i) \ge F_\Delta(x, n);$$

hence $F_{\Delta_i}(x, n+N) \ge F_\Delta(x, n)$, where $N = \max_i N_i$ and

$$(k-1)F_\Delta(x, n) \le \sum_{i=1}^{k-1} F_{\Delta_i}(x, n+N).$$

Since N is fixed, we let $n \to \infty$ to obtain

$$(k-1)\overline{f}_x(\Delta) \le \overline{f}_x(\cup_{i=1}^{k-1}\Delta_i) = 1.$$

□

Proof of Proposition 3.3.

For an arc Δ and $\epsilon > 0$ find k and an arc $\Delta' \supset \Delta$ of length $l/k < l(\Delta) + \epsilon$. Then

$$\overline{f}_x(\Delta) < \overline{f}_x(\Delta') < \frac{l}{k-1} < (l(\Delta) + \epsilon)\frac{k}{k-1}.$$

Letting $\epsilon \to 0$ and then $k \to \infty$ gives $\overline{f}_x(\Delta) \leq l(\Delta)$. Combined with

$$\overline{f}_x(A) := \limsup_{n\to\infty} \frac{F_A(x,n)}{n} = 1 - \liminf_{n\to\infty} \frac{F_{A^c}(x,n)}{n} = 1 - \underline{f}_x(A^c).$$

for $A = \Delta^c$, this also gives $\underline{f}_x(\Delta) \geq l(\Delta)$. This proves that the limit exists and equals $l(\Delta)$. □

- We call

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

  the *characteristic function* of A.

- We define

$$F_A(x,n) := \sum_{k=0}^{n-1} \chi_A(R_\alpha^k(x)),$$

  and accordingly the relative frequency is $\sum_{k=0}^{n-1} \chi_A(R_\alpha^k(x))/n$.

- By the definition of the integral, $l(\Delta) = \int_{S^1} \chi_\Delta(\phi)d\phi$, Proposition 3.3 can be reformulated as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(R_\alpha^k(x)) = \int_{S^1} \chi_\Delta(\phi)d\phi.$$

Definition 4.1

The *Birkhoff averaging operator* $\mathcal{B}_n$ is the operator that associates to a function φ the function $\mathcal{B}_n(\varphi) := \sum_{k=0}^{n-1} \varphi \circ R_\alpha^k / n$ given by

$$\mathcal{B}_n(\varphi)(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(R_\alpha^k(x)).$$

- $\mathcal{B}_n$ is linear: $\mathcal{B}_n(a\varphi + b\psi) = a\mathcal{B}_n(\varphi) + b\mathcal{B}(\psi)$.
- $\mathcal{B}_n$ is nonnegative: If $\varphi \geq 0$, then $\mathcal{B}_n(\varphi) \geq 0$.
- $\mathcal{B}_n$ is nonexpanding: $\sup_{x\in S^1} \mathcal{B}_n(\varphi)(x) \leq \sup_{x\in S^1} \varphi(x)$.
- $\mathcal{B}_n$ preserves the average: $\int_{S^1} \mathcal{B}_n(\varphi)(x)dx = \int_{S^1} \varphi(x)dx$.

Proposition 4.2

- For any step function φ that is a linear combination of characteristic functions of arcs, $\lim_{n\to\infty} \mathcal{B}_n(\varphi) = \int_{S^1} \varphi(x)dx$.
- For any function φ that is a uniform limit of step functions we also have $\lim_{n\to\infty} \mathcal{B}_n(\varphi) = \int_{S^1} \varphi(x)dx$.

Lemma 4.3

Every continuous function is the uniform limit of step functions, as is every function with finitely many discontinuity points and with one-sided limits at these points (piecewise continuous functions).

Proposition 4.4

If α is irrational and φ is continuous, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(R_\alpha^k(x)) = \int_{S^1} \varphi(x)dx$$

Theorem 4.5

If α is irrational and φ is Riemann integrable, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(R_\alpha^k(x)) = \int_{S^1}\varphi(x)dx$$

Proof.

- Pick a partition ofS^1 into a finite number of arcs I_i.
- The corresponding lower and upper Riemann sums $\sum_i \min \varphi|_{I_i} l(I_i)$ and $\sum_i \max \varphi|_{I_i} l(I_i)$ can be interpreted as integrals of steps function $\varphi_1 = \min \varphi|_{I_i}$ on I_i and $\varphi_2 = \max \varphi|_{I_i}$ on I_i. By definition of Riemann integrability, the partition can be chosen such that

$$\int_{S^1} \varphi(\phi) d\phi - \epsilon \leq \int_{S^1} \varphi_1(\phi) d\phi \leq \int_{S^1} \varphi_2(\phi) d\phi \leq \int_{S^1} \varphi(\phi) d\phi + \epsilon.$$

- This implies that

$$\int_{S^1} \varphi(\phi) d\phi - \epsilon \leq \int_{S^1} \varphi_1(\phi) d\phi = \lim_{n\to\infty} \mathcal{B}_n(\varphi_1) \leq \liminf_{n\to\infty} \mathcal{B}_n(\varphi)$$

$$\leq \limsup_{n\to\infty} \mathcal{B}_n(\varphi) \leq \lim_{n\to\infty} \mathcal{B}_n(\varphi_2) = \int_{S^1} \varphi_2(\phi) d\phi \leq \int_{S^1} \varphi(\phi) d\phi + \epsilon.$$

Letting $\epsilon \to 0$ gives the result.

Remark 4.6

The condition of Riemmann integrability is essential. To see this, take a point x_0 and define the set A as the union of the arcs of length 2^{-k+2} centered at $R_\alpha^k(x_0)$ for $k \geq 0$. Although some of these arcs overlap, A is a union of arcs the sum of whose lengths is less than $1/2$, whereas $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (R_\alpha^k(x)) = 1$. Of course, χ_A is not Riemann integrable.

Definition 4.7

Given a function φ, we call

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(R_\alpha^k(x))$$

its *time average* as sampled by following the orbit of x under the iterates of the rotation R_α. The integral $\int_{S^1} \varphi(x)dx$ is called the *space average* of the function φ.

Definition 4.8

If X is a compact metric space and $f : X \to X$ a continuous map, then f is said to be *uniquely ergodic* if

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k(x))$$

converges to a constant uniformly (in x) for every continuous function φ.

Alternate oroof of Proposition 4.4.

Define the characters $c_m(x) := e^{2\pi imx} = \cos 2\pi mx + i\sin 2\pi mx$. If $m \neq 0$, then

$$c_m(R_\alpha(x)) = e^{2\pi im(x+\alpha)} = e^{2\pi im\alpha}e^{2\pi imx} = e^{2\pi im\alpha}c_m(x)$$

and

$$|\frac{1}{n}\sum_{k=0}^{n-1} c_m(R_\alpha^k(x))| = |\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi imk\alpha}| = \frac{|1-e^{2\pi imn\alpha}|}{n|1-e^{2\pi im\alpha}|} \leq \frac{2}{n|1-e^{2\pi im\alpha}|} \to 0$$

as $n \to \infty$.

Note that Birkhoff operators are linear, so if $p(x) = \sum_{i=-l}^{l} a_i c_i(x)$ is a trigonometric polynomial, then $\lim_{n\to\infty} \mathcal{B}_n(p)(x)$ exists and is constant. It is a_0 as this constant is the integral of p over S^1. The same argument as above allow us to pass to uniform limits of trigonometric polynomials, that is, all continuous functions. □

- The circle is a compact abelian group, and the rotation can be represented in group terms as the group multiplication or translation

$$L_{g_0} : G \to G,\ L_{g_0} g = g_0 g.$$

- The orbit of the unit element $e \in G$ is the cyclic subgroup $\{g_0^n\}_{n \in \mathbb{Z}}$.
- Theorem 2.1 is closely related to the fact that the circle does not have proper infinite closed subgroups.
- To say that an orbit is dense requires a notion of approximation, so we define a *topological group* to be a group with a metric for which every L_g is a homeomorphism and taking inverses is continuous.

Proposition 6.1

If the translation L_{g_0} on a topological group G is transitive, then it is minimal.

Proof.

For $g, g' \in G$ denote by $A, A' \subset G$ the closures of the orbits of g and g', respectively. Now $g_0^n g' = g_0^n g(g^{-1}g')$, so $A' = Ag^{-1}g'$ and $A' = G$ if and only if $A = G$. □