Lecture 11: Invertible Circle Maps

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Proposition 1.1

If $f : S^1 \to S^1$ is continuous, then there exists a continuous $F : \mathbb{R} \to \mathbb{R}$, called a *lift* of f to $\mathbb{R}$, such that

$$f \circ \pi = \pi \circ F,$$

that is, $f([z]) = [F(z)]$. Such a lift is unique up to an additive integer constant, and $\deg(f) := F(x+1) - F(x)$ is an integer independent of $x \in \mathbb{R}$ and the lift F. It is called the *degree* of f. If f is a homeomorphism, then $|\deg(f)| = 1$.

Proof.

Existence: Pick a point $p \in S^1$. Then $p = [x_0]$ for some $x_0 \in \mathbb{R}$ and $f(p) = [y_0]$ for some $y_0 \in \mathbb{R}$. From these choices of x_0 and y_0 define $F : \mathbb{R} \to \mathbb{R}$ by requiring that $F(x_0) = y_0$, F is continuous, and $f([z]) = [F(z)]$ for all $z \in \mathbb{R}$. One can construct such an F by varying the initial point p continuously, which causes $f(p)$ to vary continuously. Then there is no ambiguity of how to vary x and y continuously, and thus $F(x) = y$ defines a continuous map.

To elaborate, take $\delta > 0$ such that $d([x], [x']) \leq \delta$ implies $d(f([x]), f([x'])) < 1/2$. Then defines F on $[x_0 - \delta, x_0 + \delta]$ as follows: If $|x - x_0| \leq \delta$, then $d(f([x]), q) < 1/2$ and there is a unique $y \in (y_0 - 1/2, y_0 + 1/2)$ such that $[y] = f([x])$. Define $F(x) = y$. Analogous steps extend the domain by another δ at a time, until F is defined on an interval of unit length. Then $f([z]) = [F(z)]$ defines F on $\mathbb{R}$.

Proof.

Uniqueness: Suppose $\tilde{F}$ is another lift. Then $[\tilde{F}(x)] = f([x]) = [F(x)]$ for all x, meaning $\tilde{F} - F$ is always an integer. Because it is continuous it must be constant.

Degree: $F(x+1) - F(x)$ is an integer (now evidently independent of the choice lift) because $[F(x+1)] = f([x+1]) = f([x]) = [F(x)]$. By continuity $F(x+1) - F(x) =: \deg(f)$ must be constant.

Invertibility: If $\deg(f) = 0$, then $F(x+1) = F(x)$ and thus F is not monotone. Then f is noninvertible because it cannot be monotone. If $|\deg(f)| > 1$, then $|F(x+1) - F(x)| > 1$ and, by the Intermediate-Value Theorem, there exists $y \in (x, x+1)$ with $|F(y) - F(x)| = 1$. Then $f([y]) = f([x])$ and $[y] \neq [x]$, so f is noninvertible. □

Definition 1.2

Suppose f is invertible. If $\deg(f) = 1$, then we say that f is orientation-preserving; if $\deg(f) = -1$, then f is said to reverse orientation.

Remark 1.3

The function $F(x) - x\deg(f)$ is periodic because

$$F(x+1) - (x+1)\deg(f) = F(x) + \deg(f) - (x+1)\deg(f) = F(x) - x\deg(f)$$

for all x.

Lemma 1.4

If f is an orientation-preserving circle homeomorphism and F a lift, then

$$F(y) - y \leq F(x) - x + 1$$

for all $x, y \in \mathbb{R}$.

Proof.

Let $k = \lfloor y - x \rfloor$. Then

$$\begin{aligned} F(y) - y &= F(y) + F(x+k) - F(x+k) + (x+k) - (x+k) - y \\ &= (F(x+k) - (x+k)) + (F(y) - F(x+k)) - (y - (x+k)). \end{aligned}$$

Now $F(x+k) - (x+k) = F(x) - x$ and $0 \leq y - (x+k) < 1$ by choice of k, so $F(y) - F(x+k) \leq 1$. Thus the right-hand side above is at most $F(x) - x + 1 - 0$. □

Proposition 2.1

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism and $F : \mathbb{R} \to \mathbb{R}$ a *lift* of f. Then

$$\rho(F) := \lim_{|n|\to\infty} \frac{1}{n}(F^n(x) - x)$$

exists for all $x \in \mathbb{R}$. The number $\rho(F)$ is independent of x and well defined up to an integer; that is, if $\tilde{F}$ is another lift of f, then $\rho(F) - \rho(\tilde{F}) = F - \tilde{F} \in \mathbb{Z}$. This number $\rho(F)$ is called the *rotation number* of f and it is rational if and only if f has a periodic point.

Definition 2.2

A sequence $(a_n)_{n\in\mathbb{N}}$ with $a_{n+m} \leq a_n + a_m$ is said to be *subadditive*.

Lemma 2.3

If a sequence $(a_n)_{n\in\mathbb{N}}$ satisfies $a_{n+m} \le a_n + a_{m+k} + L$ for all $m, n \in \mathbb{N}$ and some k and L, then $\lim_{n\to\infty} a_n/n \in \mathbb{R} \cup \{-\infty\}$ exists.

Proof.

The condition $a_{m+k} \leq a_m + a_{2k} + L$ gives

$$a_{m+n} \leq a_m + a_n + a_{2k} + 2L = a_m + a_n + L',$$

so we may take $k = 0$. Let $a := \liminf_{n\to\infty} a_n/n \in \mathbb{R} \cup \{-\infty\}$.

If $a < b < c$ and $n > 2L/(c-b)$ such that $a_n/n < b$, then for any $l \geq n$ that satisfies $l(c-b) > 2\max_{r<n} a_r$ we can write $l = nk + r$ with $r < n$. This implies

$$\begin{aligned} a_l/l &\leq (ka_n + a_r + kL)/l \\ &\leq a_n/n + a_r/l + (L/n) < c, \end{aligned}$$

so $\limsup_{l\to\infty} a_l/l \leq c$. Since $c > a$ was arbitrary, this proves the lemma. □

Proof of Proposition 2.1.

Independence of x: We know that $F(x+1) = F(x) + 1$. If $|x-y| < 1$, then $|F(x) - F(y)| < 1$ and

$$\begin{aligned} &\left|\frac{1}{n}|F^n(x) - x| - \frac{1}{n}|F^n(y) - y|\right| \\ \leq\ & \frac{1}{n}(|F^n(x) - F^n(y)| + |x - y|) \\ \leq\ & \frac{2}{n}. \end{aligned}$$

Thus the rotation numbers of x and y coincide, if one of them exists.

Proof of Proposition 2.1 (Cont.)

Existence: Take $x \in \mathbb{R}$ and $a_n := F^n(x) - x$. Then

$$\begin{aligned} a_{m+n} &= F^{m+n}(x) - x \\ &= F^m(F^n(x)) - F^n(x) + a_n \\ &\leq a_m + 1 + a_n. \end{aligned}$$

Thus a_n/n converges, but possibly, to $-\infty$. However,

$$\begin{aligned} \frac{a_n}{n} &= \frac{1}{n}\sum_{i=0}^{n-1}(F^{i+1}(x) - F^i(x)) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}(F(F^i(x)) - F^i(x)) \\ &\geq \min(F(y) - y), \end{aligned}$$

so the limit is a real number $\rho(F)$. Also, $\rho(F+m)$ is equal to $\lim_{|n|\to\infty}(1/n)(F^n(x) + nm - x) = \rho(F) + m$ for $m \in \mathbb{Z}$, so $\rho(F)$ is well defined $\pmod 1$.

Proof of Proposition 2.1 (Cont.)

Periodic points: If f has a q-periodic point, then $F^q(x) = x + p$ for a lift F of it and some $p \in \mathbb{Z}$. If $m \in \mathbb{N}$, then

$$\frac{F^{mq}(x) - x}{mq} = \frac{1}{mq} \sum_{i=0}^{m-1} F^q(F^{iq}(x)) - F^{iq}(x) = \frac{mp}{mq} = \frac{p}{q};$$

so $\rho(F) = p/q$.

Conversely, for any lift F the definition of rotation number yields

$$\rho(F^m) = \lim_{n\to\infty} \frac{1}{n}((F^m)^n(x) - x) = m \lim_{n\to\infty} \frac{1}{mn}(F^{mn}(x) - x) = m\rho(F);$$

so if $\rho(f) = p/q \in \mathbb{Q}$, then $\rho(f^q) = 0$ since the rotation number is defined up to an integer, Therefore we need only show:

Claim If $\rho(f) = 0$, then f has a fixed point.

Proof of Proposition 2.1 (Cont.)

Suppose f has no fixed point and let F be a lift such that $F(0) \in [0,1)$. Then $F(x) - x \notin \mathbb{Z}$ for all $x \in \mathbb{R}$. Therefore, $0 < F(x) - x < 1$ for all $x \in \mathbb{R}$. Since $F - \mathrm{Id}$ is continuous and periodic, it attains its minimum and maximum and therefore there exists a $\delta > 0$ such that

$$0 < \delta \leq F(x) - x \leq 1 - \delta < 1$$

for all $x \in \mathbb{R}$. In particular, we can take $x = F^i(0)$ and use

$$F^n(0) = F^n(0) - 0 = \sum_{i=0}^{n-1} (F^{i+1}(0) - F^i(0))$$

to get

$$n\delta \leq F^n(0) \leq (1-\delta)n$$

or

$$\delta \leq \frac{F^n(0)}{n} \leq 1 - \delta.$$

As $n \to \infty$, this gives $\rho(F) \neq 0$, which proves the claim by contraposition. □

Proposition 2.4

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. Then all periodic orbits have the same period.

In fact, if $\rho(f) = [p/q]$ with $p, q \in \mathbb{Z}$ relatively prime, then the lift F of f, with $\rho(F) = p/q$ satisfies $F^q(x) = x + p$ whenever $[x]$ is a periodic point, that is, the set of periodic points of f lifts to the set of fixed points $F^q - \mathrm{Id} - p$.

Proof.

If $[x]$ is a periodic point, then $F^r(x) = x + s$ for some $r, s \in \mathbb{Z}$ and

$$\frac{p}{q} = \rho(F) = \lim_{n\to\infty} \frac{F^{nr}(x) - x}{nr} = \lim_{n\to\infty} \frac{ns}{nr} = \frac{s}{r}.$$

This means that $s = mp$ and $r = mq$ and that therefore $f^{mq}(x) = x + mp$.

Claim. $F^q(x) = x + p$.

If $F^q(x) - p > x$, then monotonicity of F implies

$$F^{2q}(x) - 2p = F^q(F^q(x) - p) - p \geq F^q(x) - p > x$$

and inductively $F^{mq}(x) - mp > x$, which is impossible. Likewise, $F^q(x) - p < x$ is impossible because it implies $F^{mq}(x) - mp < x$. This proves the claim. □

Proposition 2.5

The rotation number depends continuously on the map in the C^0 topology.

Proposition 3.1

If $f, h : S^1 \to S^1$ are orientation-preserving homeomorphisms, then $\rho(h^{-1} f h) = \rho(f)$.

Proof.

Let F and H be lifts of f and h, respectively, that is, $\pi F = f\pi$ and $\pi H = h\pi$. Then $\pi H^{-1} = h^{-1}h\pi H^{-1} = h^{-1}\pi H H^{-1} = h^{-1}\pi$, so H^{-1} is a lift of h^{-1}. Also, $H^{-1}FH$ is a lift of $h^{-1}fh$ since

$$\pi H^{-1}FH = h^{-1}\pi FH = h^{-1}f\pi H = h^{-1}fh\pi.$$

Suppose H is such that $H(0) \in [0, 1)$. We need to estimate

$$|H^{-1}F^nH(x) - F^n(x)| = |(H^{-1}FH)^n(x) - F^n(x)|.$$

(1) For $x \in [0, 1)$ we have $0 - 1 < H(x) - x < H(x) < H(1) < 2$, and by periodicity $|H(x) - x| < 2$ for $x \in \mathbb{R}$. Similarly, $|H^{-1}(x) - x| < 2$ for $x \in \mathbb{R}$.

(2) If $|y - x| < 2$, then $|F^n(y) - F^n(x)| < 3$ since $|[y] - [x]| \le 2$ and thus

$$\begin{aligned} -3 \le [y] - [x] - 1 = F^n([y]) - F^n([x] + 1) < F^n(y) - F^n(x) \\ < F^n([y] + 1) - F^n([x]) = [y] + 1 - [x] \le 3. \end{aligned}$$

Proof.

These two estimates yield

$$\begin{aligned}
&|H^{-1}F^nH(x) - F^n(x)| \\
\leq &|H^{-1}F^nH(x) - F^nH(x)| + |F^nH(x) - F^n(x)| \\
< &2 + 3,
\end{aligned}$$

so $|H^{-1}F^nH(x) - F^n(x)|/n < 5/n$ and so $\rho(H^{-1}FH) = \rho(F)$. □

Definition 4.1

Given $x_0, x_1, \cdots, x_{n-1} \in S^1$, take $\tilde{x}_0, \cdots, \tilde{x}_{n-1} \in [\tilde{x}_0, \tilde{x}_0 + 1) \subset \mathbb{R}$ such that $[\tilde{x}_i] = x_i$. Then the *ordering* of $(x_0, \cdots, x_{n-1})$ on S^1 is the permutation σ of $\{1, \cdots, n-1\}$. Such that $\tilde{x}_0 < \tilde{x}_{\sigma(1)} < \cdots < \tilde{x}_{\sigma(n-1)} < \tilde{x}_0 + 1$.

Proposition 4.2

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) = [p/q]$. Suppose p and q are relatively prime and there is an $x \in S^1$ such that $f^q(x) = x$. Then the ordering of $\{x, f(x), f^2(x), \cdots, f^{q-1}(x)\}$ on S^1 is given by $\sigma(i) = ki \pmod q$, where $kp \equiv 1 \pmod q$.

Proof.

Fix $\tilde{x} \in \pi^{-1}([x])$ and a lift F of f such that $F^q(\tilde{x}) = \tilde{x} + p$. Then $[\tilde{x}, \tilde{x} + p]$ is partitioned (up to common endpoints) into $p \cdot q$ subintervals by $A := \pi^{-1}(\{x, f(x), f^2(x), \cdots, f^{q-1}(x)\})$, and into q subintervals $I_i = [F^i(\tilde{x}), F^{i+1}(\tilde{x})]$, $i = 0, \cdots, q-1$. Since F is a bijection between any I_i and I_{i+1} and preserves A, each I_i contains $p+1$ points of A. Take $k, r \in \mathbb{Z}$ such that the right neighbor of $\tilde{x}$ in A is $\tilde{x}_1 = F^k(\tilde{x}) - r$. Since $\overline{F} = F^k - r$ is increasing on $\mathbb{R}$ and preserves A, the facts that $\tilde{x}_1 = \overline{F}(\tilde{x})$ is the nearest right neighbor of $\tilde{x}$ in A and that $[\tilde{x}, F(\tilde{x})]$ is divided into p subintervals by A show that $\overline{F}^p(\tilde{x}) = F(\tilde{x})$ and hence $f^{kp}(x) = f(x)$. Therefore k is the unique integer between 0 and $q-1$ such that $kp \equiv 1 \bmod q$, and the ordering of the orbit $\{x, f(x), f^2(x), \cdots, f^{q-1}(x)\}$ is given by $ki \equiv \sigma(i) \bmod q$. □

Proposition 4.3

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with rational rotation number $\rho(f) = p/q \in \mathbb{Q}$. Then there are two possible types of nonperiodic orbits for f:

(1) If f has exactly one periodic orbit, then every other point is heteroclinic under f^q to two points on the periodic orbit. These points are different if the period is greater than one. (If the period is one, then all orbits are homoclinic to the fixed point.)

(2) If f has more than one periodic orbit, then each nonperiodic point is heteroclinic under f^q to two points on different periodic orbits.

Proposition 4.4

If $I \subset \mathbb{R}$ is a closed bounded interval and $f : I \to I$ is a non-decreasing continuous map, then all $x \in I$ are either fixed or asymptotic to a fixed point of f. If f is increasing (hence invertible), then all $x \in I$ are either fixed or heteroclinic to adjacent fixed points.

Proof of Proposition 4.3.

We can identify f^q with a homeomorphism of an interval by taking a lift z of a fixed point of f^q and restricting a lift $F^q(\cdot) - p$ of f to $[z, z+1]$. Then the statement follows from the above Proposition applied to this interval map, except for the last part of (2), that the periodic orbits in question are different. But if there is an interval $I = [a, b] \subset \mathbb{R}$ such that a and b are adjacent zeros of $F^q - \mathrm{Id} - p$ and a, b project to the same periodic orbit, then f has only one periodic orbit because, if $[a] = x \in S^1$, $[b] = f^k(x) \in S^1$, then $\bigcup_{n=0}^{q-1} f^{nk}(\pi(a, b))$ covers the complement of $\{f^n(x)\}_{n=0}^{q-1}$ in S^1 and contains no periodic points. By invariance, $f^{nk}(\pi((a, b)))$ does not either. □

Lemma 4.5

If $I \subset \mathbb{R}$ is an interval whose endpoints are adjacent zeros of $F^q - \mathrm{Id} - p$, then $F^q - \mathrm{Id} - p$ has the same sign on the interiors of I and $F(I)$.

Proof.

If $F^q - Id - p > 0$ on I, then $F^q(x) > x + p$ for all $x \in I$ and monotonicity of F implies $F^q(F(x)) = F(F^q(x)) > F(x+p) = F(x) + p$ for all $x \in I$. Therefore $F^q - Id - p > 0$ on $F(I)$.
The case $F^q - \mathrm{Id} - p < 0$ is similar. □

Thus for a circle homeomorphism with a periodic point all orbits are asymptotically periodic with the same period and in a coherent way.

Proposition 5.1

Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of an orientation-preserving homeomorphism $f : S^1 \to S^1$ with $\rho(f) := \rho(F) \notin \mathbb{Q}$. Then, for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and $x \in \mathbb{R}$,

$n_1\rho + m_1 < n_2\rho + m_2$ if and only if $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$.

Proof.

We do not have equality on the right for any x because this would imply $F^{n_1}(x) - F^{n_2}(x) \in \mathbb{Z}$, and hence that $[x]$ is periodic. Thus, for given $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, the continuous expression $F^{n_1}(x) + m_1 - F^{n_2}(x) - m_2$ never changes sign and the second inequality is independent of x.
Now assume $F^{n_1} + m_1 < F^{n_2}(x) + m_2$ for all x. Substituting $y := F^{n_2}(x)$ shows that this is equivalent to

$$F^{n_1 - n_2}(y) - y < m_2 - m_1 \text{ for all } y \in \mathbb{R}.$$

In particular, for $y = 0$ we get $F^{n_1 - n_2}(0) < m_2 - m_1$, and $y = F^{n_1 - n_2}(0)$ gives

$$F^{2(n_1 - n_2)}(0) < (m_1 - m_1) + F^{n_1 - n_2}(0) < 2(m_2 - m_1).$$

Inductively, $F^{n(n_1 - n_2)}(0) < n(m_2 - m_1)$ and

$$\rho = \lim_{n \to \infty} \frac{F^{n(n_1 - n_2)}(0)}{n(n_1 - n_2)} < \lim_{n \to \infty} \frac{n(m_2 - m_1)}{n(n_1 - n_2)} = \frac{m_2 - m_1}{n_1 - n_2}$$

(with strict inequality since $\rho \notin \mathbb{Q}$). Consequently, $n_1 \rho + m_1 < n_2 \rho + m_2$. This proves "if". Reversing all inequalities proves the converse. □

Lemma 5.2

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of S^1 without periodic points, $m, n \in \mathbb{Z}$, $m \neq n$, $x \in S^1$ a closed interval with endpoints $f^m(x)$ and $f^n(x)$. Then every semiorbit meets I.

Remark 5.3

For $x \neq y \in S^1$ there are exactly two intervals in S^1 with endpoints x and y. The lemma holds for either choice. Since x is not periodic, I is not a point.

Proof.

- Consider positive semiorbits $(f^n(y))_{n\in\mathbb{N}}$.
- The proof for negative semiorbits is exactly the same.
- To prove the lemma it suffices to show that the backward iterates of I cover S^1.
- That is, $S^1 \subset \cup_{k\in\mathbb{N}} f^{-k}(I)$.

Proof.

- Let $I_k := f^{-k(n-m)}(I)$ and note that theses are all contiguous: If $k \in \mathbb{N}$, then I_k and I_{k-1} have a common endpoint.
- Consequently, if $S^1 \neq \bigcup_{k\in\mathbb{N}} I_k$, then the sequence of endpoints converge to some $z \in S^1$.
- But then

$$\begin{aligned} z &= \lim_{k\to\infty} f^{-k(n-m)}(f^m(x)) = \lim_{k\to\infty} f^{(-k+1)(n-m)}(f^m(x)) \\ &= \lim_{k\to\infty} f^{(n-m)} f^{-k(n-m)}(f^m(x)) = f^{(n-m)}(\lim_{k\to\infty} f^{-k(n-m)}(f^m(x))) \\ &= f^{(n-m)}(z) \end{aligned}$$

is periodic, contrary to the assumption.

□

If there are periodic points, they provide all the accumulation points of orbits. Now we see what set plays this role when the rotation number is irrational.

Definition 5.4

The set $\omega(x) := \cap_{n\in\mathbb{N}}\overline{\{f^i(x)|i \geq n\}}$ of accumulation points of the positive semiorbit of x is called the ω-*limit* set of x.

Proposition 5.5

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of S^1 without periodic points. Then $\omega(x)$ is independent of x and $E := \omega(x)$ is perfect and either S^1 or nowhere dense.

Note that perfect nowhere dense sets are Cantor sets, that is, they are homeomorphic to the standard middle-third Cantor set. Therefore, this result produces Cantor sets directly from the dynamics of a circle map – at least when we give an example where this is the possibility that is actually realized.

Independent of x

Proof.

- We first show that $\omega(x) = \omega(y)$ for $x, y \in S^1$.
- Let $z \in \omega(x)$. Then there is a sequence l_n in $\mathbb{N}$ such that $f^{l_n}(x) \to z$.
- If $y \in S^1$, then there exist $k_m \in \mathbb{N}$ such that $f^{k_m}(y) \in I_m := [f^{l_m}(x), f^{l_{m+1}}(x)]$.
- But then $\lim_{m\to\infty} f^{k_m}(y) = z$ and thus $z \in \omega(y)$.
- Therefore $\omega(x) \subset \omega(y)$ for all $x, y \in S^1$.
- By symmetry $\omega(x) = \omega(y)$ for all $x, y \in S^1$.

$E := \omega(x)$ is either S^1 or nowhere dense

Proof.

- We first how that E is the smallest closed nonempty f-invariant set.
- If $\emptyset \neq A \subset S^1$ is closed and $x \in A$, then $\{f^k(x)\}|_{k\in\mathbb{Z}} \subset A$ since A is invariant.
- And $E = \omega(x) \subset \overline{\{f^k(x)\}|_{k\in\mathbb{Z}}} \subset A$ since A is closed.
- Thus any closed invariant set A is either empty or contains E.
- In particular, $\emptyset$ and E are the only closed invariant subsets of E itself.
- Since E is closed, it contains its boundary, which is itself a closed set.
- The boundary is also invariant because a boundary point is a point for any neighbourhood U of which we have $U \cap E \neq \emptyset$ and $U \backslash E \neq \emptyset$, a property that persists when we apply a homeomorphism.
- Therefore the boundary ∂E of E is a closed invariant subset of E and as such we must have either $\partial E = \emptyset$ and hence $E = S^1$, or else $\partial E = E$, which implies that E is nowhere dense.

E is perfect

Proof.

- Let $x \in E$.
- Since $E = \omega(x)$, there is a sequence k_n such that $\lim_{n\to\infty} f^{k_n}(x) = x$.
- Since there are no periodic orbits, $f^{n_k}(x) \neq x$ for all n.
- Consequently, x is an accumulation point of E since $f^{k_n}(x) \in E$ for all n by invariance. □

Theorem 6.1

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number ρ. Then there is a continuous monotone map $h : S^1 \to S^1$ with $h \circ f = R_\rho \circ h$.

(1) If f is transitive, then h is a homeomorphism.

(2) If f is not transitive, then h is not invertible.

Proof.

We first construct the lift of h only on the lift of a single orbit and show that it is monotone. We then extend it to the closure of that lift and, using monotonicity, “fill in” any gaps that may be left. Finally we define h as the projection.

Pick a lift $F : \mathbb{R} \to \mathbb{R}$ of f and $x \in \mathbb{R}$. Let $B := \{F^n(x) + m\}_{n,m\in\mathbb{Z}}$ be the total lift of the orbit of $[x]$. Define $H : B \to \mathbb{R}$, $F^n(x) + m \mapsto n\rho + m$, where $\rho := \rho(F)$. This map is monotone, and $H(B)$ is dense in $\mathbb{R}$.

Proof.

If we write $\tilde{R}_\rho : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + \rho$, then $H \circ F = \tilde{R}_\rho \circ H$ on B because

$$H \circ F(F^n(x) + m) = H(F^{n+1}(x) + m) = (n+1)\rho + m$$

and

$$\tilde{R}_\rho \circ H(F^n(x) + m) = \tilde{R}_\rho(n\rho + m) = (n+1)\rho + m.$$

Lemma 6.2

H has a continuous extension to the closure $\overline{B}$ of B.

Proof.

If $y \in \overline{B}$, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in B such that $y = \lim_{n\to\infty} x_n$. To show that $H(y) := \lim_{n\to\infty} H(x_n)$ exists and is independent of the choice of a sequence approximating y, observe first that the left and right limits exist and are independent of the sequence since H is monotone. If the left and right limits disagree, then $\mathbb{R}\backslash H(B)$ contains an interval, which contradicts the density of $H(B)$. □

Proof of Theorem 6.1 (Cont.)

H can now easily be extended to $\mathbb{R}$. Since $H : \overline{B} \to \mathbb{R}$ is monotone and surjective there is no choice in defining H on the intervals complementary to $\overline{B}$: Set $H =$ const. on those intervals, choosing the constant equal to the values at the endpoints. This gives a map $H : S^1 \to S^1$ since for $z \in B$ we have

$$H(z+1) = H(F^n(x) + m + 1) = n\rho + m + 1 = H(z) + 1,$$

and this property persists under continuous extension.

To decide invertibility note that in the transitive case we start from a dense orbit and so $\overline{B} = \mathbb{R}$ and h is a bijection. In the nontransitive case, H is constant on the intervals complementary to the orbit closure that we used. □