Lecture 13: Recurrence and Equidistribution in Higher Dimension

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- Consider the n-dimensional torus

$$\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}} = \mathbb{R}^n/\mathbb{Z}^n = \underbrace{\mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}}_{n \text{ times}}.$$

- A natural *fundamental domain* for $\mathbb{R}^n/\mathbb{Z}^n$ is the unit cube:

$$I^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | 0 \le x_i \le 1 \text{ for } i = 1, \cdots, n\}.$$

- This means that, to represent the torus, we identify opposite faces of I^n, that is, we identify $(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)$ with $(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$.
- These two points represent the same element of the torus.

- Similar to the case of the circle, there are two convenient coordinate systems on $\mathbb{T}^n$, namely,
  (1) multiplicative, where the elements of $\mathbb{T}^n$ are represented as $(z_1, \cdots, z_n)$ with $z_i \in \mathbb{C}$ and $|z_i| = 1$ for $i = 1, \cdots, n$; and
  (2) additive, when they are represented by n-vectors $(x_1, \cdots, x_n)$, where each coordinate is defined mod 1.
- The correspondence $(x_1, \cdots, x_n) \mapsto (e^{2\pi i x_1}, \cdots, e^{2\pi i x_n})$ establishes an isomorphism between these two representations.
- By the way, these coordinate systems are called multiplicative and additive, respectively, because there is a "group" structure on the torus that can be viewed as multiplication or as addition: For any two elements $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ there is an element $x + y$ defined by $x + y = (x_1 + y_1, \cdots, x_n + y_n)$ (in additive notation), and this addition has negatives, just like that in $\mathbb{R}^n$.
- In multiplicative notation, the same structure is defined by taking products coordinatewise, and inverses are just reciprocals.
- In fact, the additive interpretation of this structure is just addition modulo 1 and hence "inherited" from $\mathbb{R}^n$ under the identification of vectors modulo 1.

- In additive notation let $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{T}^n$.
- Consider the natural multidimensional generalization of rotations given by the translation

$$T_\gamma(x_1, \cdots, x_n) = (x_1 + \gamma_1, \cdots, x_n + \gamma_n) \pmod 1.$$

- If all coordinates of the vector γ are rational numbers, say $\gamma_i = p_i/q_i$ with relatively prime p_i and q_i for each $i = 1, \cdots, n$, then T_γ is periodic. Its minimal period is the least common multiple of the denominators $q_1, \cdots, q_n$.
- However, unlike the cases of the circle and linear flows on the 2-torus, minimality is not the only alternative to periodicity.
- For example, if $n = 2$ and $\gamma = (\alpha, 0)$, where α is an irrational number, then the torus $\mathbb{T}^2$ splits into a family of invariant circles $x_2 =$ const., and every orbit stays on one of these circles and fills it densely.

Definition 2.1

A set $A \subset \mathbb{R}$ is said to be rationally independent if $x_1, \cdots, x_n \in A$ and $(k_1, \cdots, k_n) \in \mathbb{Z}^{n+1} \backslash \{0\}$ imply $\sum_{i=1}^{n} k_i \gamma_i \neq 0$.

Rational independence of $\gamma_1, \cdots, \gamma_n$ and 1 means that $k_0 + \sum_{i=1}^{n} k_i \gamma_i \neq 0$ for $(k_0, k_1, \cdots, k_n) \in \mathbb{Z}^{n+1} \backslash \{0\}$. Equivalently, $\sum_{i=1}^{n} k_i \gamma_i$ is not an integer for any collection of integers $k_1, \cdots, k_n$, except for $k_1 = k_2 = \cdots = k_n = 0$.

Note that in the case of a single number this is exactly irrationality.

Proposition 2.2

The translation T_γ on $\mathbb{T}^2$ is minimal if and only if the numbers γ_1, γ_2, and 1 are rationally independent, that is, there are no two nonzero integers k_1, k_2 such that $k_1\gamma_1 + k_2\gamma_2 \in \mathbb{Z}$.

- The reason for the requirement on the translation vector is not so hard to see.
- We saw that a linear $(T_\gamma^t)_{t\in\mathbb{R}}$ on the 2-torus is minimal if its translation vector γ has irrational slope.
- Therefore, a translation T_γ can only be minimal if $\gamma_1/\gamma_2 \notin \mathbb{Q}$ or $k_1\gamma_1 + k_2\gamma_2 \neq 0$ for any $(k_1, k_2) \in \mathbb{Z}^2\setminus\{0\}$.
- On the other hand, this is not quite sufficient, because if γ_1 is rational. say, then the first coordinate of any orbit can only take infinitely many values and the orbit cannot be dense.
- To rule out such problems requires the minimality condition above.

- We have seen linear flows on the 2-torus.
- On the n-torus linear flows are likewise given as a one-parameter group of translations

$$T_\omega^t(x_1, \cdots, x_n) = (x_1 + t\omega_1, \cdots, x_n + t\omega_n) \pmod 1.$$

- Since the flow $\{T_\omega^t\}$ is minimal if for some t_0 the transformation $T_\omega^{t_0}$ is minimal, we can establish the criterion for minimality for this case.

Proposition 3.1

The flow $\{T^t_\omega\}$ is minimal if and only if the numbers $\omega_1, \cdots, \omega_n$ are rationally independent.

Proof.

Since $T^t_\omega = T_{t\omega}$, minimality follows from Proposition 2.2 once we find $t \in \mathbb{R}$ such that $\sum_{i=1}^n tk_i\omega_i \notin \mathbb{Z}$ for any nonzero integer vector $(k_1, \cdots, k_n)$. To this end note that if $k \in \mathbb{Z}$ then $s \sum_{i=1}^n k_i\omega_i = k$ implies $s = k/\sum_{i=1}^n k_i\omega_i$. Only countable many such s's arise, so any $t \in \mathbb{R}\backslash\{k/\sum_{k_i} \omega_i | k_1, \cdots, k_n, k \in \mathbb{Z}, (k_1, \cdots, k_n) \neq 0\}$ as required.
On the other hand, if $\sum_{k_i} \omega_i = 0$ for some nonzero vector $(k_1, \cdots, k_n)$, then the function $\varphi(x) = \sin 2\pi(\sum_{i=1}^n k_i x_i)$ is continuous, nonconstant, and invariant under the flow $\{T^t_\omega\}$. Therefore the flow is not minimal, because $\varphi^{-1}([0,1])$ is a closed invariant set. □

- In the one-dimensional case we used arcs (intervals) as natural "windows" through which to measure the frequency of visits.
- A natural counterpart for the n-torus will be n-*parallelepipeds*, $\Delta = \Delta_1 \times \cdots \times \Delta_n$, where $\Delta_1, \cdots, \Delta_n$ are arcs.
- For $n = 2$ it is natural to call a parallelepiped a *rectangle*.
- The *volume* $\mathrm{vol}(\Delta)$ of Δ is defined as the product of the lengths of the arcs $\Delta_1, \cdots, \Delta_n$.

Definition 4.1

A sequence $(x_m)_{m\in\mathbb{N}}$ in $\mathbb{T}^n$ is said to be *uniformly distributed* if

$$\lim_{m\to\infty} \frac{\mathrm{card}\{k \in \{1, \cdots, m\} | x_k \in \Delta\}}{m} = \mathrm{vol}(\Delta)$$

for every n-parallelepiped $\Delta \subset \mathbb{T}^n$.

Theorem 4.2

If $(\gamma_1, \gamma_2, 1)$ are rationally independent, then every semiorbit of the translation $T_{(\gamma_1,\gamma_2)}$ is uniformly distributed.

Define

$$F_\Delta(x,n) := \text{card}\{k \in \mathbb{Z} | 0 \le k \le n, T_\gamma^k(x) \in \Delta\}$$

for any $x \in \mathbb{T}^2$ and any rectangle Δ.

Proposition 4.3

Consider two rectangles $\Delta = \Delta_1 \times \Delta_2$ and $\Delta' = \Delta'_1 \times \Delta'_2$ such that $l(\Delta_i) < l(\Delta'_i)$, $i = 1, 2$. There is an $N_0 \in \mathbb{N}$, which depends on Δ, Δ', and γ, such that if $x \in \mathbb{T}^2$, $N \ge N_0$, and $n \in \mathbb{N}$, then $F_{\Delta'}(x, n+N) \ge F_\Delta(x,n)$.

Proof.

By assumption there is a translation T_β of the rectangle Δ that lies inside Δ'. By minimality of T_γ we can find $N_0 \in \mathbb{N}$ such that the translation $T_\gamma^{n_0}\Delta$ is so close to $T_\beta\Delta$ that $T_\gamma^{N_0}\Delta \subset \Delta'$. Thus $T_\gamma^n(x) \in \Delta$ implies $T_\gamma^{n+N_0}(x) \in \Delta'$ and $F'_\Delta(x, n+N) \geq F'_\Delta(x, n+N_0) \geq F_\Delta(x, n)$ for $n \geq N_0$. □

Proof of Theorem 4.2.

- Similarly to the one-dimensional case, take a rectangle $\Delta = \Delta_1 \times \Delta_2$, where $l(\Delta_1) = l(\Delta_2) = 1/k$.
- Divide the torus $\mathbb{T}^2$ into $(k-1)^2$ disjoint rectangles, each being the product of two arcs of length $1/(k-1)$, and apply Proposition 4.3 we can get

$$\overline{f}(\Delta) := \limsup_{n\to\infty} \frac{F_{\Delta(x,n)}}{n} \leq 1/(k-1)^2.$$

- Finally, let $\Delta = \Delta_1 \times \Delta_2$ be an arbitrary rectangle.
- Fix $\epsilon > 0$ and a rectangle $\Delta' = \Delta_1' \times \Delta_2'$ such that $\Delta_i \subset \Delta_i'$ for $i = 1, 2$; the lengths of Δ_i' are l_i/k; and vol$\Delta' \leq$ vol$\Delta + \epsilon$.

□

- By the subadditivity of $\overline{f}$ we obtain

$$\overline{f}(\Delta) \leq \overline{f}(\Delta') \leq (\frac{k}{k-1})^2 \text{vol}\Delta' < (\frac{k}{k-1})^2(\text{vol}\Delta + \epsilon).$$

- Since ϵ is arbitrarily small and k arbitrarily large, this implies that $\overline{f}(\Delta) \leq \text{vol}\Delta$ for any rectangles Δ and hence (by subadditivity of $\overline{f}$) for any finite union of disjoint rectangles.
- In particular, since $\mathbb{T}^2 \backslash \Delta$ is the union of three disjoint rectangles, this implies that

$$\underline{f}(\Delta) := \liminf_{n\to\infty} \frac{F_\Delta(x,n)}{n} = 1 - \overline{f}(\mathbb{T}^2 \backslash \Delta) \geq 1 - \text{vol}(\mathbb{T}^2 \backslash \Delta) = \text{vol}\Delta.$$

- And hence $\underline{f}(\Delta) = \overline{f}(\Delta) = \text{vol}\Delta$.

Theorem 4.4

Let $\gamma = (\gamma_1, \gamma_2)$ and φ any Riemann-integrable function on $\mathbb{T}^2$. If the numbers $1, \gamma_1, \gamma_2$ are rationally independent, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T_\gamma^k(x_1, x_2)) = \int_{\mathbb{T}^2} \varphi(\theta_1, \theta_2) d\theta_1 d\theta_2$$

uniformly in $(x_1, x_2) \in \mathbb{T}^2$.

Proof.

- The passage from uniform distribution for rectangles to uniform distribution for continuous and, more generally, Riemann-integrable functions goes exactly as in the one-dimensional case.
- If Δ is a rectangle, then

$$\text{vol}\Delta = \int_{\mathbb{T}^2} \chi_\Delta(\delta_1, \delta_2) d\delta_1 d\delta_2.$$

- And, by definition, a function φ is Riemann-integrable if for any $\epsilon > 0$ there exist finite linear combinations φ_1, φ_2 of characteristic functions of rectangles such that $\varphi_1 \leq \varphi \leq \varphi_2$ and

$$\int_{\mathbb{T}^2} \varphi_2(\theta_1, \theta_2) d\theta_1 d\theta_2 < \int_{\mathbb{T}^2} \varphi_1(\theta_1, \theta_2) d\theta_1 d\theta_2 + \epsilon.$$

- In particular, any continuous function or any bounded function with finitely many discontinuity points is Riemann integrable.

□

We use the additive notation,. Such a translation is minimal if and only if the orbit of 0 is dense, because if $x \in \mathbb{T}^2$, then

$$T_\gamma(x) = x + \gamma = 0 + \gamma + x = T_\gamma(0) + x \pmod 1;$$

that is, the orbit $\mathcal{O}(x)$ of x is $T_x(\mathcal{O}(0))$, and therefore it is dense if and only if $\mathcal{O}(0)$ is dense because T_x is a homeomorphism.

Pick $\epsilon > 0$ and consider the set D_ϵ of all iterates $T_\gamma^m(0)$ that are in the ϵ-ball $B(0, \epsilon)$ around 0. There are two possibilities:

(1) For some $\epsilon > 0$ the set D_ϵ is linearly dependent (that is, lies on a line).

(2) For any $\epsilon > 0$ the set D_ϵ contains two linearly independent vectors.

Below we prove three corresponding lemmas.

Lemma 5.1

(2) $\Rightarrow$ minimality.

Proof.

It suffices to show that the orbit of 0 is dense. Take $\epsilon > 0$ and suppose $v_1, v_2 \in D_\epsilon$ are linearly independent. This means that they span a small parallelogram $\{av_1 + bv_2 | a, b \in [0,1]\}$. The vertices of this parallelogram are all part of $\mathcal{O}(0)$: This is already known for 0, v_1 and v_2, and for $v_1 + v_2$ this is easy to see by representing v_1 and v_2 as $V_1 = 0 + m_1\gamma - k(m_1)$ and $V_2 = 0 + m_2\gamma - k(m_2)$ in $\mathbb{R}^2$, respectively, where $k(m_1)$ and $k(m_2)$ are those integer vectors for which $\|V_1\| < \epsilon$ and $\|V_2\| < \epsilon$. Then
$V_1 + V_2 = 0 + (m_1 + m_2)\gamma - (k(m_1) + k(m_2)) = T_\gamma^{m_1+m_2}(0) \pmod 1$
and hence $v_1 + v_2 = T_\gamma^{m_1+m_2}(0)$.

Proof.

Furthermore, the orbit of 0 contains all integer linear combinations of v_1 and v_2 [because $kV_1 + lV_2 = T_\gamma^{km_1+lm_2}(0) \pmod 1$]. Therefore, consider the tiling of the plane defined by the translates of $R := \{aV_1 + bV_2 | a, b \in [0,1]\}$ by integer multiples of V_1 and V_2. This covers the plane with similar parallelograms, which have only boundary points in common, and every point of the plane is within ϵ of one of the vertices of these tiles. In particular, every point of $[0,1] \times [0,1]$ is within ϵ of some vertex, that is, every point of $\mathbb{T}^2$ is within ϵ of some point of $\mathcal{O}(0)$, so $\mathcal{O}(0)$ is dense in $\mathbb{T}^2$. □

Lemma 5.2

(1) $\Rightarrow$ rational dependence

Proof.

If 0 is periodic, then γ_1 and γ_2 are rational and we are done.

From now on assume that the orbit of 0 is infinite. Then for any $\epsilon > 0$ it contains two points $p = T_\gamma^m(0)$ and $q = T_\gamma^n(0)$ such that $\|q - p\| < \epsilon$. Then there are points $P = m\gamma \in \mathbb{R}^2$ and $Q = n\gamma + k \in \mathbb{R}^2$ such that $\epsilon > \|P - Q\| = \|m\gamma - n\gamma - k\| = \|(m-n)\gamma - k\|$, which means that $T_\gamma^{m-n}(0) - k \in B(0, \epsilon)$ and $D_\epsilon \neq \{0\}$ for all $\epsilon > 0$.

If $\epsilon > 0$ is as in (1), then $\{0\} \neq D_{\epsilon'} \subset D_\epsilon$ is linearly dependent for all $\epsilon' < \epsilon$. Thus D_ϵ lies on a unique line L through 0 given by an equation $ax + by = 0$.

Proof.

Claim. $\mathcal{O}(0)$ is dense on the projection of L.

Since $D_{\epsilon'} \neq \{0\}$ for all $\epsilon' < \epsilon$, there are points $0 \neq p_{\epsilon'} \in D_{\epsilon'}$ and hence points $P = n\gamma - k \in L \cap B(0, \epsilon')$ (with $n \in F, k \in \mathbb{Z}^2$). But then $\{mP | m \in \mathbb{Z}\}$ is ϵ'-dense in L and since it projects into $\mathcal{O}(0)$ so $\mathcal{O}(0)$ is dense on the projection of L.

Proof.

Now a and b are rationally dependent because otherwise the slope of L is irrational, so the projection of L to $\mathbb{T}^2$ is dense and by the Claim so is $\mathcal{O}(0)$. Therefore there exists $(k_1, k_2) \in \mathbb{Z}^2 \backslash \{0\}$ such that $ak_1 - bk_2 = 0$. If $a = 0$ (or $b = 0$), then $ax + by = 0 \Leftrightarrow y = 0$ (or $x = 0$). Otherwise, multiply $ax + by = 0$ by $k_1/b = k_2/a$ to get $k_2 x + k_1 y = 0$, that is, we may take $a, b \in \mathbb{Z}$. If $n\gamma - k$ lies on the line $ax + by = 0$, then $an\gamma_1 - k_1 + bn\gamma_2 - k_2 = 0$ or $an\gamma_1 + bn\gamma_2 = k_1 + k_2$, which gives rational dependence. □

Lemma 5.3

Rational dependence $\Rightarrow$ (1).

Proof.

Suppose $k_1\gamma_1 + k_2\gamma_2 = N \in \mathbb{Z}$ and divide γ_1 to get $\gamma_2/\gamma_1 = (N - k_1)/k_2 =: s \in \mathbb{Q}$ (if $k_2 \neq 0$), that is, the iterates $(n\gamma_1, n\gamma_2)$ of 0 under repeated translation by γ lie on the line $y = sx$ with rational slope s. This projects to the torus as an orbit of the linear flow T^t_γ, which we found to be closed and hence not dense when $\gamma_2/\gamma_1 \in \mathbb{Q}$. Therefore the orbit of 0 under T_γ is not dense either, implying (1). (If $k_2 = 0$, then $k_1 \neq 0$ and the same argument works after exchanging x and y.) □

The Kronecker–Weyl method of proving uniform distribution starting from trigonometric polynomials, then proceeding to continuous functions, and finally to characteristic functions, also works in higher dimension. Again, to simplify notation we consider the two-dimensional case.

The characters are defined as group "homomorphisms" of $\mathbb{T}^2$ to S^1, where we view $\mathbb{T}^2$ as an additive group (as de- scribed at the beginning of this chapter) and S^1 is considered as the group of complex numbers of absolute value one with multiplication as the group operation.

- A homomorphism is a map that preserves this group structure, that is, the image of the sum of two elements is the product of their images.
- To be specific, if we use additive notation for the torus, then the characters have the following form:

$$\begin{aligned} c_{m_1,m_2}(x_1,x_2) &= e^{2\pi i(m_1x_1+m_2x_2)} \\ &= \cos 2\pi(m_1x_1+m_2x_2) + i\sin 2\pi(m_1x_1+m_2x_2), \end{aligned}$$

  where (m_1, m_2) is any pair of integers.
- Finite linear combinations of characters are called trigonometric polynomials because they also can be expressed as finite linear combinations of sines and cosines.
- Characters are eigenfunctions for the translation because

$$\begin{aligned} c_{m_1,m_2}(T_\gamma(x_1,x_2)) &= e^{2\pi i(m_1(x_1+\gamma_1)+m_2(x_2+\gamma_2))} \\ &= e^{2\pi i(m_1\gamma_1+m_2\gamma_2)} c_{m_1,m_2}(x_1,x_2). \end{aligned}$$

- A crucial observation for our purpose is that, since γ_1, γ_2 and 1 are rationally independent, that is, $m_1\gamma_1 + m_2\gamma_2$ is never an integer unless $m_1 = m_2 = 0$, the eigenvalue $e^{2\pi i(m_1\gamma_1+m_2\gamma_2)} \neq 1$ unless $m_1 = m_2 = 0$.
- The trivial character $c_{0,0} = 1$ is not changed by averaging.
- For the other characters we summation of the geometric series to obtain

$$
\begin{aligned}
&|\frac{1}{n}\sum_{k=0}^{n-1} c_{m_1.m_2}(T_\gamma^k(x_1,x_2))| \\
=&|\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi ik(m_1\gamma_1+m_2\gamma_2)}||c_{m_1,m_2}(x_1,x_2)| \\
=&|\frac{1-e^{2\pi in(m_1\gamma_1+m_2\gamma_2)}}{n(1-e^{2\pi i(m_1\gamma_1+m_2\gamma_2)})}| \\
\leq&\frac{2}{n(1-e^{2\pi i(m_1\gamma_1+m_2\gamma_2)})} \xrightarrow{n\to\infty} 0 = \int_{\mathbb{T}^2} c_{m_1,m_2}.
\end{aligned}
$$

- Using linearity of the integral one deduces that, for any finite linear combination φ of characters, that is, for any trigonometric polynomial, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T_\gamma^k(x_1, x_2)) = \int_{\mathbb{T}^2} \varphi.$$

- Now we can invoke a multidimensional version of the Weierstrass Approximation Theorem (a continuous function on the plane that is 1-periodic in both variables is a uniform limit of trigonometric polynomials) to deduce that the above equality holds for any continuous function.
- Finally, uniform distribution for rectangles follows exactly as in the one-dimensional case by finding continuous functions $\varphi_1 \leq \chi_\Delta \leq \varphi_2$ such that $\int(\varphi_2 - \varphi_1) < \epsilon$.

- It is is also easy to see within this frame-work that if 1, γ_1, and γ_2 are rationally dependent, then the translation T_γ is not minimal.
- If $m_1\gamma_1 + m_2\gamma_2 = k$ with $m_1, m_2, k \in \mathbb{Z}$, and $m_1^2 + m_2^2 > 0$, then $e^{2\pi i(m_1\gamma_1+m_2\gamma_2)} = 1$ and the values of the nonconstant character c_{m_1,m_2} do not change under translation.
- The use of the Kronecker–Weyl method allows us to bypass a comparatively subtle argument which was required to establish the condition for minimality. With this approach uniform distribution is deduced directly and rather straightforwardly from the rational independence of γ_1, γ_2, and 1. Also, the extension of the proof to arbitrary dimension using this method is completely routine.

- An understanding of linear maps and flows on tori provides a tool for describing the dynamics of an important class of linear systems, namely, maps with eigenvalues of absolute value one and linear differential equations with constant coefficients whose coefficient matrix has purely imaginary eigenvalues (and whose time-T-maps thus have eigenvalues of absolute value 1).
- Consider a linear map of $\mathbb{R}^{2m}$ whose eigenvalues form m distinct complex conjugate pairs $e^{\pm i_j}$.
- As before, each pair corresponds to a two-dimensional invariant subspace in which the map acts as a rotation with respect to proper coordinates.
- The eigenspace and these coordinates are obtained by taking a complex eigenvector wi and then choosing the real vectors $v_j = w_j + \bar{w}_j$ and $v_j = i(w_j - \bar{w}_j)$ as a basis.

- Doing this for each pair of eigenvalues gives a basis of R^{2m} with respect to which the map has a block diagonal matrix representation in which each block is a 2×2 block representing a rotation.
- This map then leaves invariant the sets given by the equations $x_{2j1}^2 + x_{2j}^2 = r_j^2$ for $j = 1, \cdots, m$.
- These sets are tori whose dimension depends on the number of r_j's that are zero.
- Specifically, such a torus can be parameterized by polar coordinates $x_{2j-1} = r_j \cos \varphi_j$, $x_{2j} = r_j \sin \varphi_j$, and the map then acts by rotation that shift φ_j to $\varphi_j + v_j$.
- Clearly any $r_j = 0$ reduces the dimension of the torus.

- Therefore the minimality criterion tells us that the restriction of the flow to such an invariant torus is minimal when $\{_j | r_j \neq 0\} \cup \{1\}$ is rationally independent.
- More generally, one can draw conclusions about the action of a linear map inside its neutral space E_0 when the restriction to this subspace has sufficiently many distinct eigenvalues.

- The motion of a point mass on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ without external forces is described by the second-order ordinary differential equation $\dot{x} = 0$, where x is defined modulo $\mathbb{Z}^n$.
- Alternatively we can write

$$\dot{x} = v,$$
$$\dot{x} = 0$$

  to see that the motion is along straight lines with constant speed, since v is preserved.
- This means that the n components of v are integrals (or constants) of motion.
- For any given v the motion corresponds to the linear flow T_v^t. Thus the phase space is $\mathbb{R}^n \times \mathbb{T}^n$ with dynamics described as follows: The tori $\{v\} \times \mathbb{T}^n$ are invariant and the motion on $\{v\} \times \mathbb{T}^n$ is given by $\{v\} \times T_v^t$.

- This flow is also called the geodesic flow on $\mathbb{T}^n$.
- The geodesics are the paths traced out on $\mathbb{T}^n$ by the orbits.
- They are projections of straight lines in $\mathbb{R}^n$ to $\mathbb{T}^n$.
- While for different initial velocity vectors v these curves may be variously dense, periodic, or neither, the orbits of the flow are never dense in the phase space.
- One way of studying this flow via a discrete-time dynamical system is to restrict attention to vectors with footpoint on the circle $y = 0$ and pointing upward.
- Each of these vectors defines an orbit of the flow that returns to this set.
- If α is the cotangent of the angle of such a vector, then the return map is given by $(x, \alpha) \mapsto (x + \alpha, \alpha)$.

- Consider a finite number of point particles with equal masses moving on the interval with elastic collisions among themselves and with the endpoints.
- Since the order of the particles cannot change, their positions $x_1, \cdots, x_n$ satisfy $0 \leq x_1 \leq \cdots \leq x_n \leq 1$.
- That is, the configuration space of this mechanical system is the simplex $T_n := \{(x_1, \cdots, x_n) | 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$.
- And the phase space is the space of tangent vectors with footpoints in T_n with appropriate conventions on the boundary.

- The n-dimensional analogs of the geometric considerations show that the system can be described as the motion of a single point particle bouncing off the faces of T_n with an n-dimensional analog of the reflection law "angle of incidence equals angle of reflection".
- This means that one determines the continuation of a trajectory after an impact on a face by taking the plane spanned by the incoming trajectory and the normal vector to the face and applies the two-dimensional reflection law in this plane.
- This prescription does not determine motions that involve collisions with an edge or vertex, that is, multiple or simultaneous collisions.

- The partial unfolding, which helped describe the billiard in the triangle in terms of the linear flow on the 2-torus, works here as well, with the fundamental domain being the n-dimensional cube of twice the linear size, that is, $\max|x_i| \leq 1$.
- The $n!2^n$ reflected copies of T_n tile this cube, and, in turn, the translated copies of this cube tile $\mathbb{R}^n$.
- Thus the complete unfolding of this motion on T_n produces the free particle motion on $\mathbb{R}^n$.
- After reducing this motion to the fundamental domain (the cube, which we identify with the n-torus) we obtain the free particle motion on the n-torus.
- Hence we can describe this motion in terms of the linear flow on the n-torus.

The mechanical equivalent of the geometric unfolding is the observation that, upon collision, any two particles exchange momenta and therefore one can consider only the transfer of momenta, which makes it appear as if the particles go through each other and only reverse direction at the boundary.