Lecture 15: Growth of Periodic Points

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Definition 0.1

For a map $f : X \to X$, let $P_n(f)$ be the number of periodic points of f with (not necessary minimal) period n, that is, the number of fixed points for f^n.

Consider the noninvertible map E_2 of the circle given in multiplicative notation by

$$E_2(z) = z^2,\ |z| = 1,$$

and in additive notation by

$$E_2(x) = 2x \pmod 1.$$

Proposition 1.1

$P_n(E_2) = 2^n - 1$ and periodic points for E_2 are dense in S^1.

Proof.

- If $E_2^n(z) = z$, then $z^{2^n} = z$, and $z^{2^n - 1} = 1$.
- Thus every root of unity of order $2^n - 1$ is a periodic point for E_2 of period n.
- There are exactly $2^n - 1$ of these, and they are uniformly spread over the circle with equal intervals.
- In particular, when n becomes large these intervals become small.

□

- We see from the above Proposition that a natural measure of asymptotic growth of the number of periodic points is the exponential growth rate $\rho(f)$ for the sequence $p_n(f)$:

$$p(f) = \limsup_{n\to\infty} \frac{\log_+ P_n(f)}{n},$$

where $\log_+(x) = \log(x)$ for $x \geq 1$, 0 otherwise.

- In particular, the above Proposition shows that $p(E_2) = \log 2$.
- The maps $E_m : x \mapsto mx \pmod 1$, where m is an integer of absolute value greater than one, represent a straightforward generalization of the map E_2.
- Not surprisingly, these maps also have dense sets of periodic orbits.

Lemma 1.2

$P_n(E_m) = |m^n - 1|$ and periodic points for E_m are dense.

Proof.

$z = E_m^n(z) = z^{m^n}$ has $|m^n - 1|$ solutions that are evenly spaced. □

- Another property of the maps E_m worth noticing is preservation of length similar to the property of preservation of phase volume discussed in the last lecture.
- Naturally, the length of an image of any arc increasing.
- However, if one considers the *complete preimage* of an arc Δ under E_m, one immediately sees that it consists of $|m|$ arcs of length $l(\Delta)/|m|$ each, placed along the circle at equal distances.
- The analysis in the last lecture can be extended to noninvertible volume-preserving maps, so recurrent points are dense in this situation as well.

- For $\lambda \in \mathbb{R}$, let $f_\lambda := \lambda x(1-x)$. For $0 \leq \lambda 4$, the f_λ map the unit interval $I = [0,1]$ into itself.
- The family f_λ is referred to as the *quadratic family*.
- Note that $P_n(f_\lambda) \leq 2^n$ because the nth iterate of f_λ is a polynomial of degree 2^n, and hence the equation $(f_\lambda)^n(x) = x$ has at most 2^n solutions.
- Here we consider the behavior of the quadratic family for large values of the parameter, namely, $\lambda \geq 4$.
- While for $\lambda > 4$ the interval $[0,1]$ is not preserved, the set of points that remains in that interval is still quite interesting.

Proposition 2.1

For $\lambda \geq 4$ we have $P_n(f_\lambda) = 2^n$.

Proof.

It suffices to prove the reverse inequality. To that end we use the following observation: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\Delta \subset [0,1]$ is an interval such that one endpoint is mapped to 0 and the other to 1, then by the Intermediate-Value Theorem there is a fixed point of f in Δ. Now $[0,1] \subset [f_\lambda(0), f_\lambda(1/2)]$ and $[0,1] \subset [f_\lambda(1/2), f_\lambda(1)]$, so there are intervals $\Delta_0 \subset [0,1/2]$ and $\Delta_1 \subset [1/2,1]$ whose images under f_λ are exactly $[0,1]$, giving us two fixed points for f. The nonzero fixed point is indeed in the interior of Δ_1 because the right endpoint of Δ_1 is 1 and hence is mapped to 0, so the other endpoint is mapped to 1 and therefore neither are fixed.

Proof.

Furthermore, the preimages of Δ_0 and Δ_1 under f consist of two intervals each, so there are four intervals whose images under f^2 are exactly $[0, 1]$. Each contains a fixed point of f_λ^2, again, every one except 0 being in the interior of the corresponding interval, so no two of these fixed points coincide.

Repeating this argument successively for higher iterates of f_λ we obtain 2^n intervals whose image under f_λ^n are $[0, 1]$, and each of which therefor contains at least one fixed point, giving 2^n distinct orbits of period n for f_λ. □

Definition 2.2

A continuous map defined on an interval that is increasing to the left of an interior point and decreasing thereafter is said to be *unimodal*.

Proposition 2.3

If $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) = f(1) = 0$, and there exists $c \in [0, 1]$ such that $f(c) > 1$, then $P_n(f) \geq 2^n$. If, in addition, f is unimodal and expanding (that is, $|f(x) - f(y)| > |x - y|$) on each interval of $f^{-1}((0, 1))$, then $P_n(f) = 2^n$.

Lemma 2.4

Denote by $\mathcal{M}_k$ the collection if continuous maps $f : [0,1] \to \mathbb{R}$ such that $f^{-1}((0,1)) = \cup_{i=1}^{k} I_i$ with $I_i \subset [0,1]$ open intervals, f monotonic on I_i, and $f(I_i) = (0,1)$. Then $f \circ g \in \mathcal{M}_{kl}$ whenever $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_l$.

Proof.

The lemma shows that $P_n(f) \geq k^n$ for $f \in \mathscr{M}_k$ because $f^n \in \mathscr{M}_{k^n}$. If f is expanding on every interval of $f^{-1}((0,1))$, then the same holds for iterates of f. This shows that on each of those intervals there is at most one solution of $f^n(x) = x$. Therefore, $P_n(f) \leq k^n$, proving equality. □

Definition 3.1

A continuously differentiable map $f : S^1 \to S^1$ is said to be an *expanding* map if $|f'(x)| > 1$ for all $x \in S^1$.

Since f' is continuous and periodic, the minimum of $|f'|$ ia attained and hence is greater than 1.

Lemma 3.2

If $f, g : S^1 \to S^1$ are continuous, then $\deg(g \circ f) = \deg(f)\deg(g)$, in particular $\deg(f^n) = \deg(f)^n$.

Proof.

If F, G are lifts of f and g, respectively, then
$G(s+k) = G(s+k-1) + \deg(g) = \cdots = G(s) + k\deg(g)$ and
$G(F(s+1)) = G(F(s) + \deg(f)) = G(F(s)) + \deg(g)\deg(f)$. □

Proposition 3.3

If $f : S^1 \to S^1$ is an expanding map, then $|\deg(f)| > 1$ and $P_n(f) = |\deg(f)^n - 1|$.

Proof.

$|f'| > 1$ implies $|F'| > 1$ for any lift, so by the Mean-Value Theorem, $|\deg(f)| = |F(x+1) - F(x)| > 1$. By the chain rule an iterate of an expanding map is itself expanding, so it suffices to consider the case $n = 1$. Take a lift F of f and consider it on the interval $[0, 1]$. The fixed points of f are the projections of the points x for which $F(x) - x \in \mathbb{Z}$. □

- The previous examples were all one-dimensional, but the patterns of the growth and distribution of periodic points observed in those examples also appear in higher dimension.
- A convenient model to demonstrate this is built from the following linear map of $\mathbb{R}^2$:

$$L(x,y) = (2x+y, x+y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- If two vectors (x,y) and (x',y') represent the same element of $\mathbb{T}^2$, that is, if $(x-x', y-y') \in \mathbb{Z}^2$, then $L(x,y) - L(x',y') \in \mathbb{Z}^2$, so $L(x,y)$ and $L(x',y')$ also represent the same element of $\mathbb{T}^2$.
- Thus L defines a map $F_L : \mathbb{T}^2 \to \mathbb{T}^2$:

$$F_L(x,y) = (2x+y, x+y) \pmod 1.$$

- The map F_L is, in fact, an automorphism of the torus viewed as an additive group.
- It is invertible because the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has determinant one.
- So L^{-1} also has integer entries and hence defines a map $F_{L^{-1}} = F_L^{-1}$ on $\mathbb{T}^2$ by the same argument.
- The eigenvalues of L are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} > 1 \text{ and } \lambda_1^{-1} = \lambda_2 = \frac{3-\sqrt{5}}{2} < 1.$$

Proposition 3.4

Periodic points of F_L are dense and $P_n(F_L) = \lambda_1^n + \lambda_1^{-n} - 2$.

Proof.

- To obtain density we show that points with rational coordinates are periodic points.
- Let $x, y \in \mathbb{Q}$.
- Taking the common denominator write $x = s/q$, $y = t/q$, where $s, t, q \in \mathbb{Z}$.
- Then

$$F_L(s/q, t/q) = ((2s + t)/q, (s + t)/q)$$

is a rational point whose coordinates also have denominator q.

Proof.

- But there are only q^2 different points on $\mathbb{T}^2$ whose coordinates can be represented as rational numbers with denominator q, and all iterates $F_L^n(s/q, t/q)$, $n = 0, 1, 2, \cdots$, belong to a finite set.
- Thus they must repeat, that is, $F_L^n(s/q, t/q) = F_L^m(s/q, t/q)$ for some $n, m \in \mathbb{Z}$. But since F_L is invertible, $F_L^{n-m}(s/q, t/q) = (s/q, t/q)$ and $(s/q, t/q)$ is a periodic point, as required.
- This gives density.

Proof.

- Now we show that points with rational coordinates are the only periodic points for F_L. Write $F_L^n(x,y) = (ax+by, cx+dy) \pmod 1$, where $a,b,c,d \in \mathbb{Z}$.
- If $F_L^n(x,y) = (x,y)$, then

$$ax + by = x + k,$$
$$cx + dy = y + l$$

  for $k, l \in \mathbb{Z}$.
- Since 1 is not an eigenvalue for L^n, we can solve for (x,y):

$$x = \frac{(d-1)k - bl}{(a-1)(d-1) - cb}, \; \frac{(a-1)l - ck}{(a-1)(d-1) - cb}.$$

- Thus $x, y \in \mathbb{Q}$.

Proof.

- Now we calculate $P_n(F_L)$.
- The map

$$G = F_L^n - \mathsf{Id} : (x, y) \mapsto ((a-1)x + by, cx + (d-1)y) \pmod 1$$

is a well-defined noninvertible map of the torus onto itself.
- As before, if $F_L^n(x, y) = (x, y)$, then $(a-1)x + by$ and $cx + (d-1)y$ are integers; hence $G(x, y) = 0 \pmod 1$, that is, the fixed points of F_L^n are exactly the preimages of the point $(0, 0)$ under G.
- Modulo 1 these are exactly the points of $\mathbb{Z}^2$ in $(L^n - \mathsf{Id})([0, 1] \times [0, 1])$. We presently show that their number is given by the area of $(L^n - \mathsf{Id})([0, 1) \times [0, 1))$, which is $|\det(L^n - \mathsf{Id})| = |(\lambda_1^n - 1)(\lambda_1^{-n} - 1)| = \lambda_1^n + \lambda_1^{-n} - 2$.

□

Lemma 3.5

The area of a parallelogram with integer vertices is the number of lattice points it contains, where points on the edges are counted as half, and all vertices count as a single point.

Proof.

- Denote the area of the parallelogram by A.
- Adding the number of lattice points it contains in the prescribed way gives an integer N, which is the same for any translate of the parallelogram.
- Now consider the canonical tiling of the plane by copies of this parallelogram translated by integer multiples of the edges.
- Denote by l the longest diagonal.
- The area of the tiles can be determined in a backward way by determining how many tiles lie in the square $[0, n) \times [0, n)$ for $n > 2l$.

Proof.

- Those that lie inside cover the smaller square $[l, n-l) \times [l, n-l)$ completely, so there are at least

$$\frac{(n-2l)^2}{A} \geq \frac{n^2}{A}(1 - \frac{4l}{n}).$$

- Since any tile intersects the square is contained in $[-l, n+l) \times [-l, n+l)$, there are at most

$$\frac{(n+2l)^2}{A} = \frac{n^2}{A}(1 + \frac{4l}{n}(1 + \frac{l}{n})) < \frac{n^2}{A}(1 + \frac{6l}{n}).$$

Proof.

- The number n^2 of integer points in the square is at least the number of points in tiles in the square and at most the number of points in tiles that intersect the square.
- Therefore

$$N \cdot \frac{n^2}{A}(1 - \frac{4l}{n}) \leq n^2 \leq N \cdot \frac{n^2}{A}(1 + \frac{6l}{n}) \text{ and } 1 - \frac{4l}{n} \leq \frac{A}{N} \leq 1 + \frac{6l}{n}$$

for all $n > 2l$.

- This shows that $N = A$.

□

Definition 3.6

If X is a metric space and $f : X \to X$ continuous, then the *inverse limit* is defined on the space

$$X' := \{(x_n)_{n\in\mathbb{Z}} | x_n \in X \text{ and } f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{Z}\}$$

by $F((x_n)_{n\in\mathbb{Z}}) := (x_{n+1})_{n\in\mathbb{Z}}$.

Consider explicitly $f = E_2$ on S^1. Then the inverse limit is the space

$$\mathbb{S} := \{(x_n)_{n\in\mathbb{Z}} | x_n \in X \text{ and } f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{Z}\}$$

with the map $F((x_n)_{n\in\mathbb{Z}}) := (x_{n+1})_{n\in\mathbb{Z}} = (2x_n)_{n\in\mathbb{Z}} \pmod 1$. This is called the *solenoid*.