Lecture 16: Topological Transitivity and Chaos

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Definition 0.1

A continuous map $f : X \to X$ of a metric space is said to be *chaotic* if it is topologically transitive and its periodic points are dense.

Proposition 1.1

Let X be a complete separable (that is, there is a countable dense subset) metric space with no isolated points. If $f : X \to X$ is a continuous map, then the following four conditions are equivalent:

(1) f is topologically transitive, that is, it has a dense orbit.

(2) f has a dense positive smeiorbit.

(3) If $\emptyset \neq U, V \subset X$, then there exists an $N \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

(4) If $\emptyset \neq U, V \subset X$, then there exists an $N \in \mathbb{N}$ such that $F^n(U) \cap V \neq \emptyset$.

Lemma 1.2

Let X be a metric space and $f : X \to X$ a continuous map. Then (1) implies (3). If X has no isolated points, then (1) implies (4). If X is separable, then (3) implies (1) and (4) implies (2).

Proof.

Let f be topologically transitive and suppose the orbit $x \in X$ is dense. Then there exists an $n \in \mathbb{Z}$ such that $f^n(x) \in U$, and there is an $m \in \mathbb{Z}$ such that $f^m(x) \in V$; hence $f^{m-n}(U) \cap V \neq \emptyset$. This implies (3).

Proof.

If we can choose $m > n$, then by taking $N := m - n$ we have even established (4). Otherwise we use the assumption that X has no isolated points, so $f^m(x)$ is not an isolated point and therefore there are $n_k \in \mathbb{Z}$ such that $|n_k| \to \infty$, $f^{n_k}(x) \in V$, and $f^{n_k} \to f^m(x)$ as $k \to \infty$. Indeed, $n_k \to -\infty$ since $n_k \leq n$ by assumption (otherwise we are in first case), so we can choose an $m' < 2m - n$ from among the n_k such that $f^{m'}(x) \in f^{m-n}(U)$. Then $x' := f^{n-m}(U)$. Then $x' := f^{n-m}(f^{m'}(x)) \in U$ and $f^{2m-n-m'}(x') = f^m(x) \in V$, so $f^N(U) \cap V \neq \emptyset$ with $N := 2m - n - m' \in \mathbb{N}$. Thus $(1) \Rightarrow (4)$ if X has no isolated points.

Proof.

Now assume separability and one of the intersection conditions (3) and (4). We give one argument to prove both that (3) implies (1) and (4) implies (2). For a countable dense subset $S \subset X$, let $U_1, U_2, \cdots$ be the countable collection of balls centered at points of S with rational radius. We need to construct an orbit or semiorbit that intersects every U_n. By (3) there exists $N_1 \in \mathbb{Z}$ such that $f^{N_1}(U_1) \cap U_2 \neq \emptyset$. If (4) holds, we can take $N_1 \in \mathbb{N}$. Let V_1 be an open ball of radius at most $1/2$ such that $\overline{V_1} \subset U_1 \cap f^{-N_1}(U_2)$. There exists $N_2 \in \mathbb{Z}$ such that $f^{N_2}(V_1) \cap U_3$ is nonempty, and if (4) holds, we can take $N_2 \in \mathbb{N}$. Again, take an open ball V_2 of radius at most $1/4$ such that $\overline{V_2} \subset V_1 \cap f^{-N_2}(U_3)$. By induction, we construct a nested sequence of open balls V_n of radii at most 2^{-n} such that $\overline{V_{n+1}} \subset V_n \cap f^{-N_{n+1}}(U_{n+2})$. The centers of these balls form a Cauchy sequence whose limit x is the unique point in the intersection $V = \cap_{n=1}^{\infty} \overline{V_n} = \cap_{n=1}^{\infty} V_n$. Then $f^{N_{n-1}}(x) \in U_n$ for every $n \in \mathbb{N}$, and all $N_n \in \mathbb{N}$ if (4) holds.

Proof.

If f is noninvertible, the last step may involve choices for negative values of N_n. Take i_k such that $N_{i_k} < 0$ for all k and $N_{i_{k+1}} < N_{i_k}$. Choose $x_0 = x$ and $x_{N_{i_k}} \in U_{i_k+1}$. Together with $f(x_k) = x_{k+1}$, this defines an orbit of x. □

Corollary 1.3

A continuous open map f of a complete separable metric space without isolated points is topologically transitive if and only if there are no two disjoint open nonempty f-invariant sets.

Definition 2.1

A continuous map $f : X \to X$ is said to be *topologically mixing* if for any two nonempty open sets $U, V \subset X$ there is an $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n > N$.

Lemma 2.2

Isometries are not topologically mixing.

Proposition 3.1

Expanding maps of S^1 are topologically mixing.

Corollary 3.2

Linear expanding maps of S^1 are chaotic.

Proposition 4.1

The automorphism F_L is topologically mixing.

Corollary 4.2

The automorphism F_L is chaotic.

Definition 5.1

A map $f : X \to X$ of a metric space is said to exhibit *sensitive dependence* on initial conditions if there is a $\Delta > 0$, called a sensitivity constant, such that for every $x \in X$ and $\epsilon > 0$ there exists a point $y \in X$ with $d(x, y) < \epsilon$ and $d(f^N(x), f^N(y)) \geq \Delta$ for some $N \in \mathbb{N}$.

Theorem 5.2

Chaotic maps exhibit sensitive dependence on initial conditions, except when the entire space consists of a single periodic orbit.

Remark 5.3

There are maps exhibiting sensitive dependence that are not chaotic, such as the linear twist $T : S^1 \times [0,1] \to S^1 \times [0,1]$ where $T(x,y) = (x+y,y)$. Here, any point x has arbitrarily nearby points (on a vertical segment through x) that moves a considerable distance away after sufficient many iterates. The set of periodic points consists of those points whose second coordinate is rational and is hence dense. On the other hand, this map is clearly not topologically transitive.

Proposition 5.4

A topologically mixing map (on a space with more than one point) has sensitive dependence.