Lecture 17: Codings

December 6, 2023

1. Linear Expanding Maps
2. Implications of Coding
   - Proof of transitivity via coding
   - Exotic asymtotics
3. Sequence Spaces
4. Coding
5. Conjugacy and Factor
6. Dynamics of Shifts and Topological Markov Chains

- The linear expanding maps

$$E_m : S^1 \to S^1, E_m(x) = mx \pmod 1$$

are chaotic.

- That is, they exhibit coexistence of dense orbits with a countable dense set of periodic orbits.
- Thus the orbit structure is both complicated and highly nonuniform.
- Now we look at these maps from a different point of view, which in turn gives a deeper appreciation of just how complicated their orbit structure really is.

- To simplify notation, assume as before that $m = 2$.
- Consider the binary intervals

$$\Delta_n^k := [\frac{k}{2^n}, \frac{k+1}{2^n}] \text{ for } n = 1, \cdots \text{ and } k = 0, 1, \cdots, 2^n - 1.$$

- Let $x = 0.x_1x_2\cdots$ be the binary representation of $x \in [0, 1]$.
- Then $2x = x_1.x_2x_3\cdots = 0.x_2x_3\cdots \pmod 1$.
- Thus $E_2(x) = 0.x_2x_3\cdots \pmod 1$.
- This is the first and easiest example of coding, which we will discuss in greater detail shortly.

This representation can be used to give another proof of topological transitivity by describing explicitly the binary representation of a number whose orbit under the iterates of E_2 is dense.

Proposition 2.1

There exists a point $x \in S^1$ such that the closure of its orbit with respect to the map E_3 in additive notation coincides with the standard middle-third Cantor set K. In particular, K is E_3-invariant and contains a dense orbit.

Proof.

The middle-third Cantor set K can be described as the set of all points on the unit interval that have a representation in base 3 with only 0's and 2's as digits. The map E_3 acts the shift of digits to the left in base 3 representation. This implies that K is E_3-invariant. It remains to show that E_3 has a dense orbit in K.

Proof.

Every point in K has a unique representation in base 3 without 1's. Let $x \in K$ and

$$0.x_1 x_2 x_3 \cdots$$

be such a representation. Let $h(x)$ be the number whose representation in base 2 is

$$0.\frac{x_1}{2}\frac{x_2}{2}\frac{x_3}{2}\cdots .$$

Thus we have constructed a map $h : K \to [0,1]$ that is continuous, nondecreasing [that is, $x > y$ implies $h(x) \geq h(y)$], and one-to-one, expect for the fact that binary rationals have two preimages each. Furthermore, $h \circ E_3 = E_2 \circ h$. Let $D \subset [0,1]$ be a dense set of points that does not contain binary rationals. Then $h^{-1}(D)$ is dense in K because, if Δ is an open interval such that $\Delta \cap K \neq \emptyset$, then $h(\Delta)$ is a nonempty interval open, closed, or semiclosed and hence contains points of D. Now take any $x \in [0,1]$ whose E_2-orbit is dense; the E_3-orbit of $h^{-1}(x) \in K$ is dense in K.

Denote by Ω_N^R the space of sequences $\omega = (\omega_i)_{i=0}^{\infty}$ whose entries are integers between 0 and $N-1$. Define a metric by

$$d_\lambda(\omega, \omega') := \sum_{i=0}^{\infty} \frac{\delta(\omega_i, \omega_i')}{\lambda^i},$$

where $\delta(k,l) = 1$ if $k \neq l$, $\delta(k,k) = 0$, and $\lambda > 2$. The same definition can be made for two-sided sequences by summing over $i \in \mathbb{Z}$:

$$d_\lambda(\omega, \omega') := \sum_{i\in\mathbb{Z}} \frac{\delta(\omega_i, \omega_i')}{\lambda^{|i|}},$$

for some $\lambda > 3$.

Consider the symmetric cylinder defined by

$$C_{\alpha_{1-n},\cdots,\alpha_{n-1}} := \{\omega \in \Omega_N | \omega_i = \alpha_i \text{ for } |i| < n\}.$$

Fix a sequence $\alpha \in C_{\alpha_{1-n},\cdots,\alpha_{n-1}}$. If $\omega \in C_{\alpha_{1-n},\cdots,\alpha_{n-1}}$, then

$$\begin{aligned} d_\lambda(\alpha,\omega) &= \sum_{i\in\mathbb{Z}} \frac{\delta(\alpha_i,\omega_i)}{\lambda^{|i|}} = \sum_{|i|\geq n} \frac{\delta(\alpha_i,\omega_i)}{\lambda^{|i|}} \\ &\leq \sum_{|i|\geq n} \frac{1}{\lambda^{|i|}} = \frac{1}{\lambda^{n-1}} \frac{2}{\lambda - 1} < \frac{1}{\lambda^{n-1}}. \end{aligned}$$

Thus $C_{\alpha_{1-n}\cdots\alpha_{n-1}} \subset B_{d_\lambda}(\alpha, \lambda^{1-n})$, the λ^{1-n}-ball around α.

If $\omega \notin C_{\alpha_{1-n} \cdots \alpha_{n-1}}$, then

$$d_\lambda(\alpha, \omega) = \sum_{i \in \mathbb{Z}} \frac{\delta(\alpha_i, \omega_i)}{\lambda^{|i|}} \geq \lambda^{1-n}$$

because $\omega_i \neq \alpha_i$ for some $|i| < n$. Thus $\omega \notin B_{d_\lambda}(\alpha, \lambda^{1-n})$, and the symmetric cylinder is the ball of radius λ^{1-n} around any of its points:

$$C_{\alpha_{1-n} \cdots \alpha_{n-1}} = B_{d_\lambda}(\alpha, \lambda^{1-n}).$$

For one-sided sequences this discussion works along the same lines.

$$\sigma : \Omega_N \to \Omega_N, \ (\sigma\omega)_i = \omega_{i+1}$$
$$\sigma^R : \Omega_N^R \to \Omega_N^R, \ (\sigma^R\omega)_i = \omega_{i+1}.$$

Definition 3.1

Let $A = (a_{ij})_{i,j=0}^{N-1}$ be an $N \times N$ matrix whose entries a_{ij} are either 0's or 1's. (We call such matrix a $0-1$ matrix.) Let

$$\Omega_A := \{\omega \in \Omega_N | a_{\omega_n \omega_{n+1}} = 1 \text{ for} n \in \mathbb{Z}\}.$$

The space Ω_A is closed and shift-invariant, and the restriction

$$\sigma_N|_{\Omega_A} =: \sigma_A$$

is called the *topological Markov chain* determined by A.

Sequence representing a given point of the phase space are called the *codes* of that point.

Definition 5.1

Suppose that $g : X \to X$ and $f : Y \to Y$ are maps of metric spaces X and Y and that there is a continuous surjective map $h : X \to Y$ such that $h \circ g = f \circ h$. Then f is said to be a *factor* of g via the *smeiconjugacy* or *factor map* h. If this h is a homeomorphism, then f and g are said to be a *conjugacy*.

Proposition 6.1

Periodic points for the shifts σ_N and σ_N^R are dense in Ω_N and Ω_N^R, correspondingly, $P_n(\sigma_N) = P_n(\sigma_N^R) = N^n$, and both σ_N and σ_N^R are topologically mixing.

- There is a useful geometric representation of topological Markov chains.
- Connect i with j by an arrow if $a_{ij} = 1$ to obtain a *Markov graph* G_A with N vertices and several oriented edges.
- We say that a finite or infinite sequence of vertices of G_A is an admissible path or admissable sequence if any two consecutive vertices in the sequence are connected by an oriented arrow.
- A point of Ω_A corresponds to a doubly infinite path in G_A with marked origin; the topological Markov chain σ_A corresponds to moving the origin to the next vertex.

Lemma 6.2

For every $i, j \in \{0, 1, \cdots, N_1\}$, the number N_{ij}^m of admissible paths of length $m+1$ that begin at x_i and end at x_j is equal to the entry a_{ij}^m of the matrix A^m.

Corollary 6.3

$P_n(\sigma_A) = \mathrm{tr} A^n$.

Proposition 6.4

$p(\sigma_A) = r(A)$, where $r(A)$ is the spectral radius.

Definition 6.5

A matrix A is said to be positive if all its entries are positive. A $0-1$ matrix A is said to transitive if A^m is positive for some $m \in \mathbb{N}$. A topological Markov chain σ_A is said to be transitive if A is a transitive matrix.

Lemma 6.6

If A^m is positive, then so is A^n for any $n \geq m$.

Lemma 6.7

If A is transitive and $\alpha_{-k}, \cdots, \alpha_k$ is admissible, that is, $a_{\alpha_i \alpha_{i+1}} = 1$ for $i = -k, \cdots, k-1$, then the intersection $\Omega_A \cap C_{\alpha_{-k}, \cdots, \alpha_k} =: C_{\alpha_{-k}, \cdots, \alpha_k, A}$ is nonempty and moreover contains a periodic point.

Proposition 6.8

If A is a transitive matrix, then the topological Markov chain σ_A is topologically mixing and its periodic orbits are dense in Ω_A; in particular, σ_A is chaotic and hence has sensitive dependence on initial conditions.